Tight Bounds for Asynchronous Randomized Consensus

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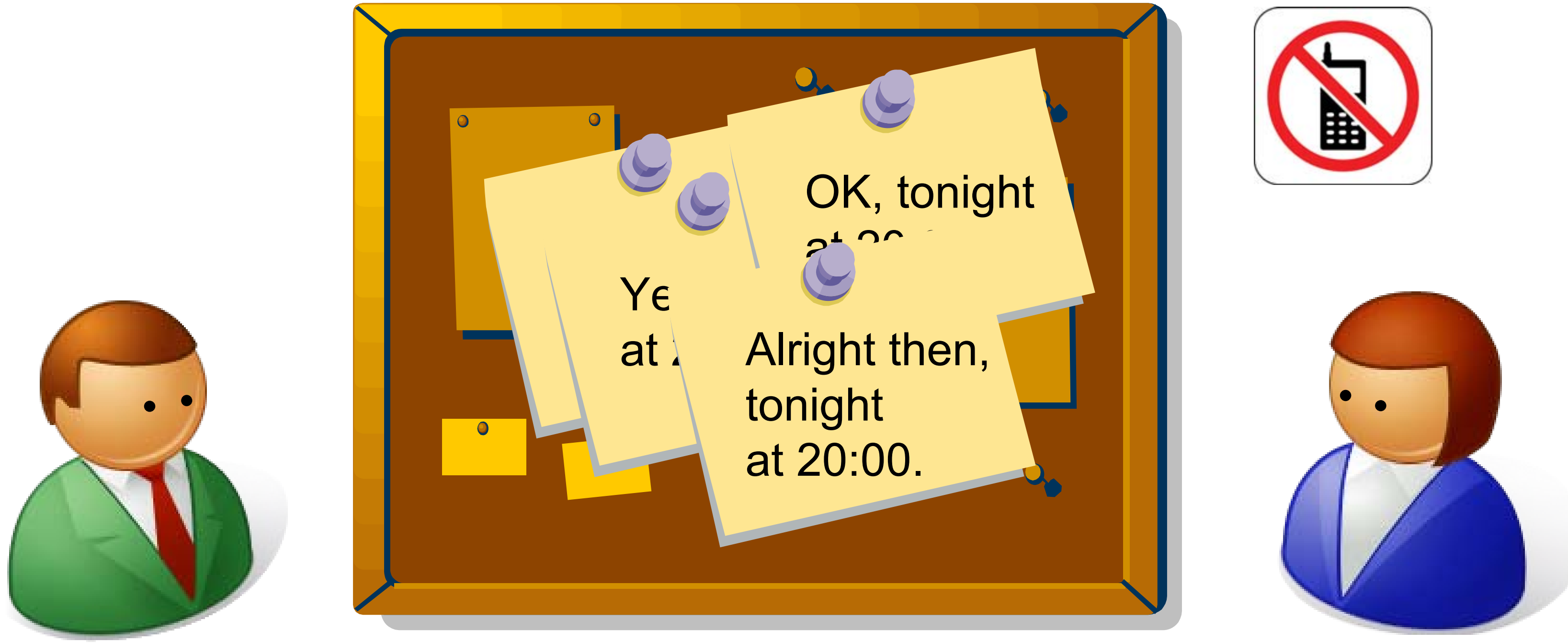
Consensus



Alice and Bob want to meet for dinner.

They write and read notes on the bulletin board.

Deterministic asynchronous consensus is impossible
[Fischer,Lynch & Paterson 1985]

Consensus

Each process p_i has an input value $x_i \in \{0,1\}$
and should produce an output value $y_i \in \{0,1\}$

- **Agreement**: all the outputs are equal
  - for every y_i,y_j that are assigned, $y_i = y_j$
- **Validity**: the output is the input of some process
  - for every y_i that is assigned, $y_i = x_j$ for some j
- **Termination**:
  every nonfaulty process eventually decides
  - every nonfaulty process p_i eventually assigns y_i

Randomized Consensus

Each process p_i has an input value $x_i \in \{0,1\}$
and should produce an output value $y_i \in \{0,1\}$

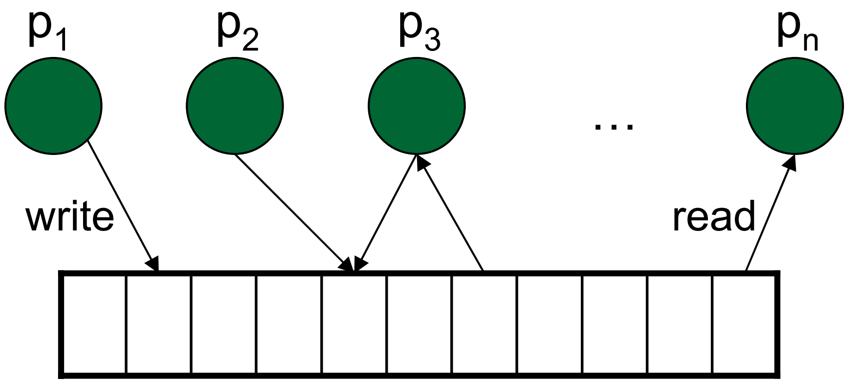
- **Agreement**: all the outputs are equal
  - for every y_i, y_j that are assigned, $y_i = y_j$
- **Validity**: the output is the input of some process
  - for every y_i that is assigned, $y_i = x_j$ for some j
- **Termination**: **with probability 1,**
  every nonfaulty process eventually decides
  - every nonfaulty process p_i eventually assigns y_i

The **total step complexity** is the **expected** total number of steps taken by all the processes

Shared Memory Multiprocessor

n asynchronous processes

Multi-writer multi-reader shared registers



- At most **f crash failures** (at some point the process stops taking steps)
  - ⇨ Cannot tell whether a process has failed or is just slow
- The ordering of steps is determined by a **strong adversary**
  - Makes the scheduling decision after observing the local coin-flips

Bounds on the Total Step Complexity

- $O(\exp(n))$ [Abrahamson 1988]
- $O(n^4)$ [Aspnes & Herlihy 1990, Aspnes 1993]
- $O(n^3)$ [Saks, Shavit & Woll 1991]
- $O(n^2 \log n)$ [Bracha & Rachman 1991]
- $\mathbf{O(n^2)}$
- $\mathbf{\Theta(n^2)}$
- $\mathbf{\Omega(n^2)}$
- $\Omega(n^2/\log^2 n)$ [Aspnes 1998]

Upper Bound: Shared Coin

Shared coin with **agreement parameter δ**: for $v \in \{0,1\}$, the probability that all processes return v is at least δ

⇨ Randomized consesnsus with step complexity $\delta^{-1}T$ where T is the step comptexity of the shared coin

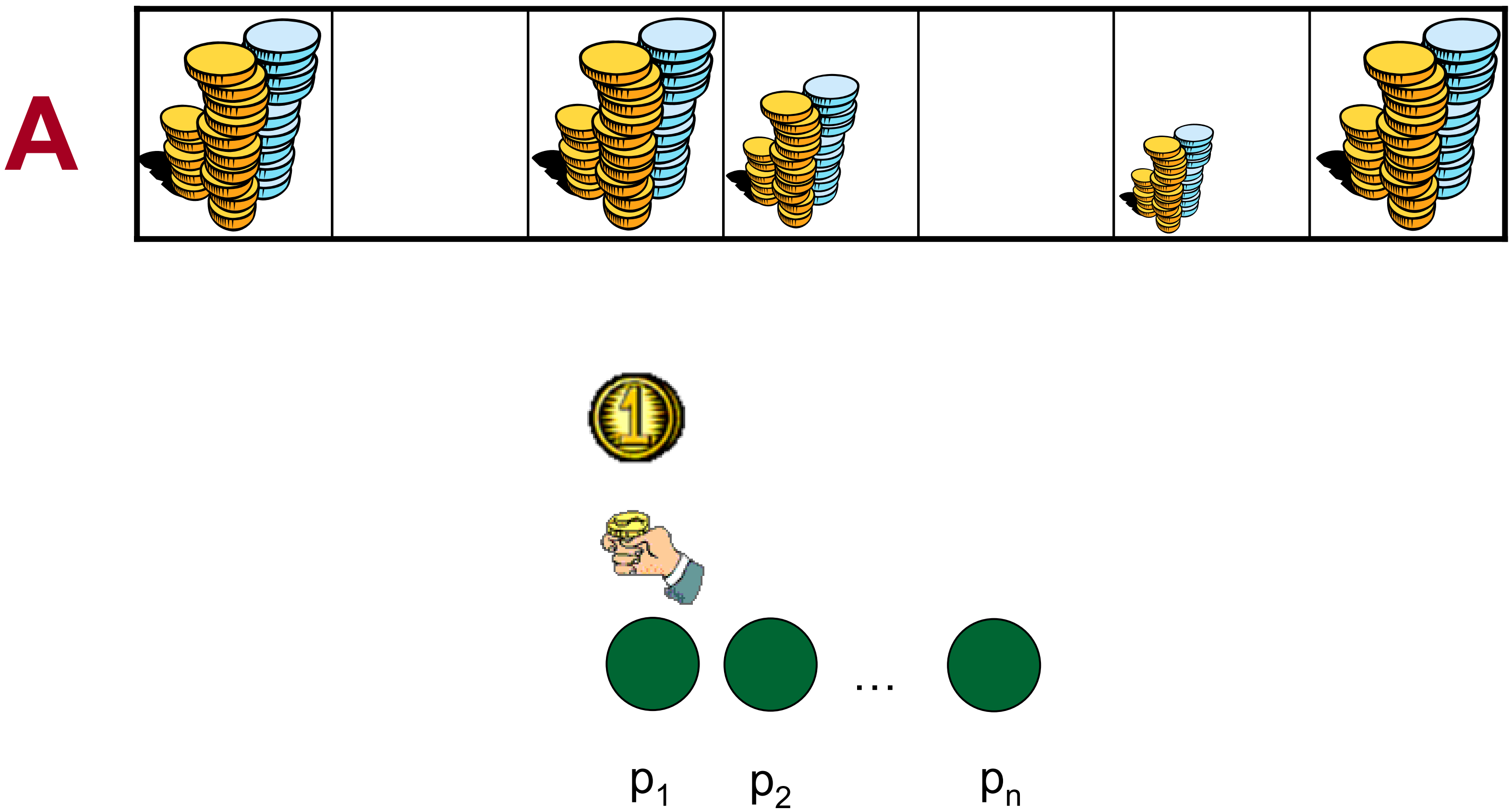
[Aspnes & Herlihy 1990]

Naïve shared coin: every process flips its own coin, agreement parameter is $1/2^n$ [Abrahamson 1988]

☞ We present a shared coin with constant agreement parameter and **$O(n^2)$** total step complexity

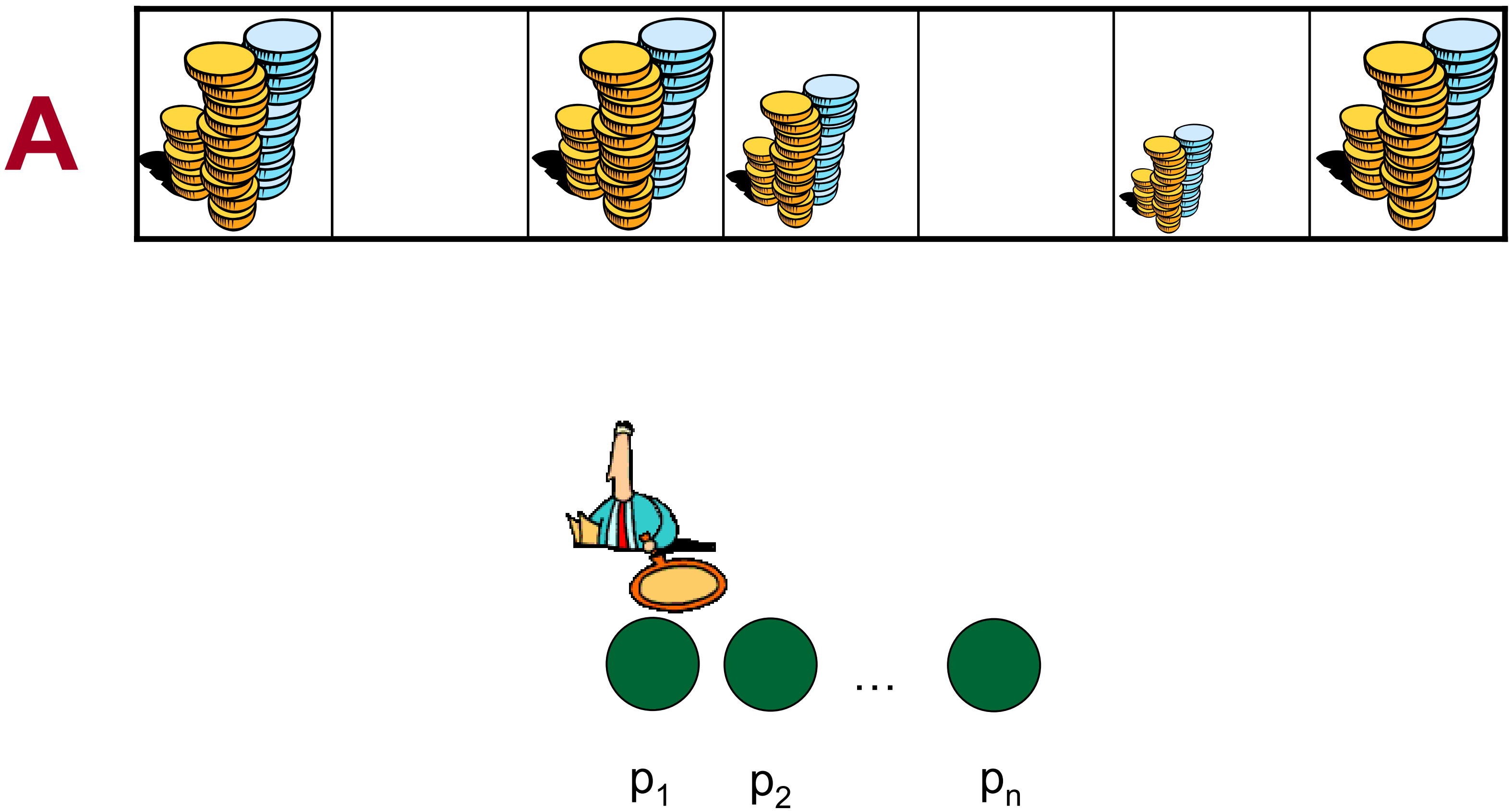
Shared Coin: Flipping Many Coins

Stop flipping coins when there are enough



Shared Coin: How Many Coins?

Collect to check how many coins were flipped

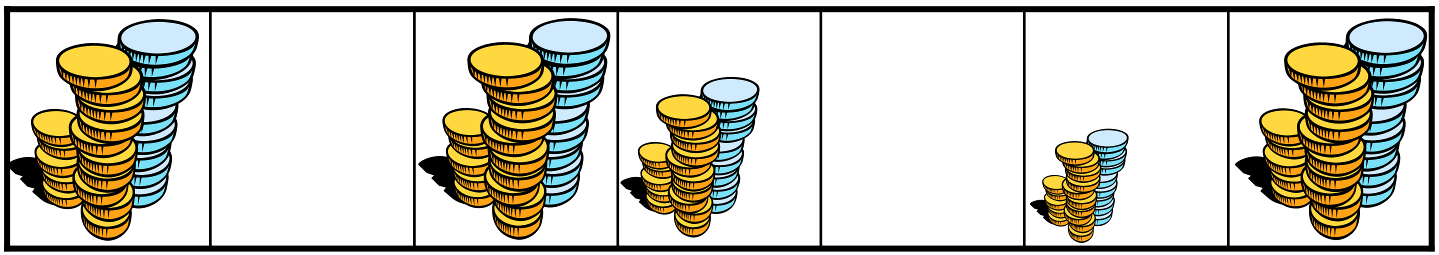


Shared Coin: Balancing Act

Collect to check how many coins were flipped

[Bracha & Rachman 1991]

A

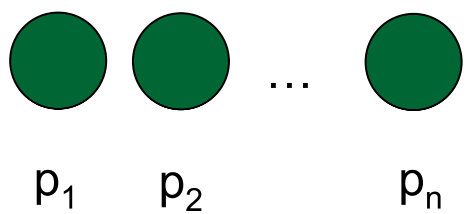


collect costs **n** steps

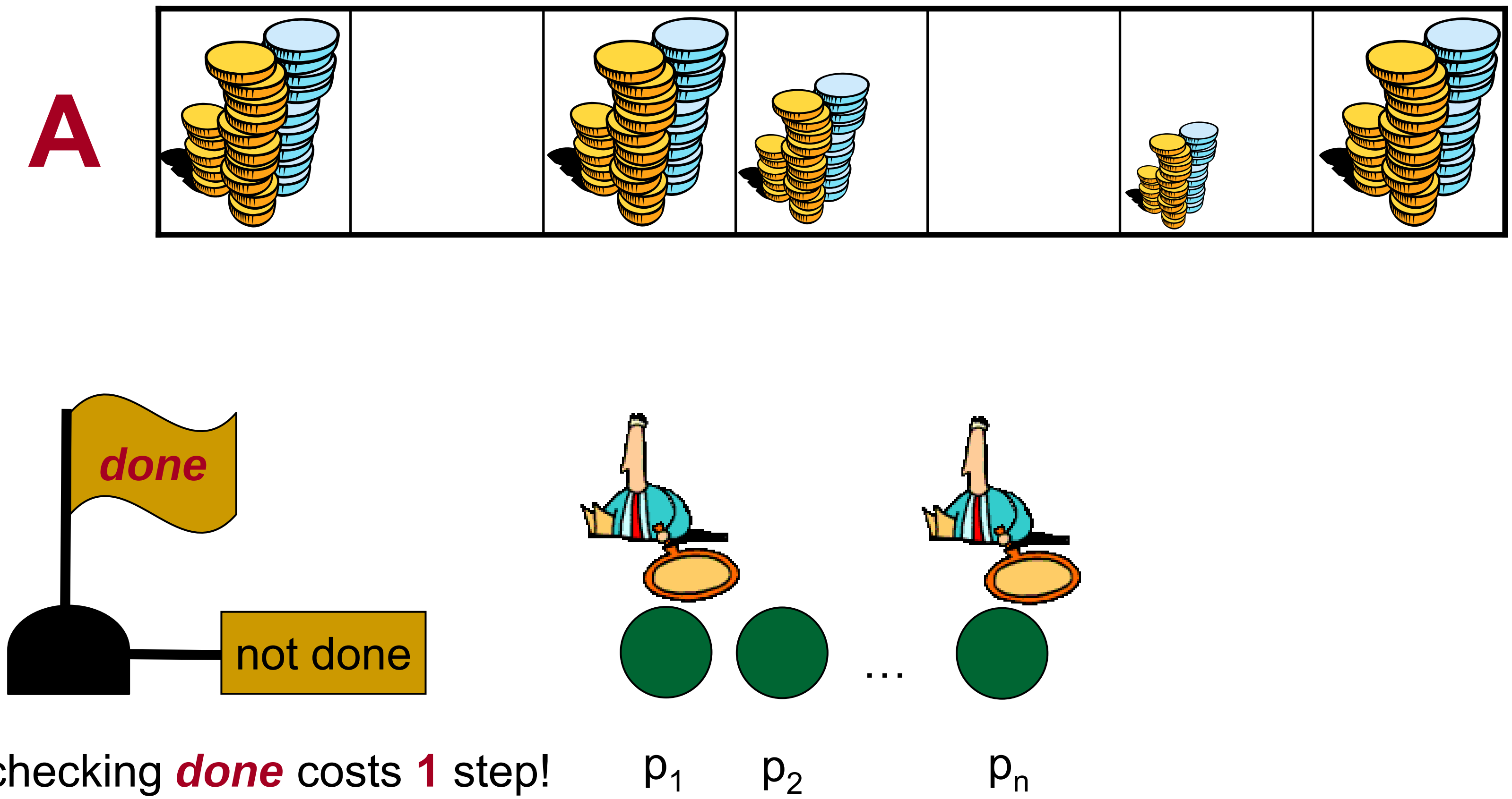
⇨ perform **fewer** collects

more coins written between collects

⇨ perform **more** collects



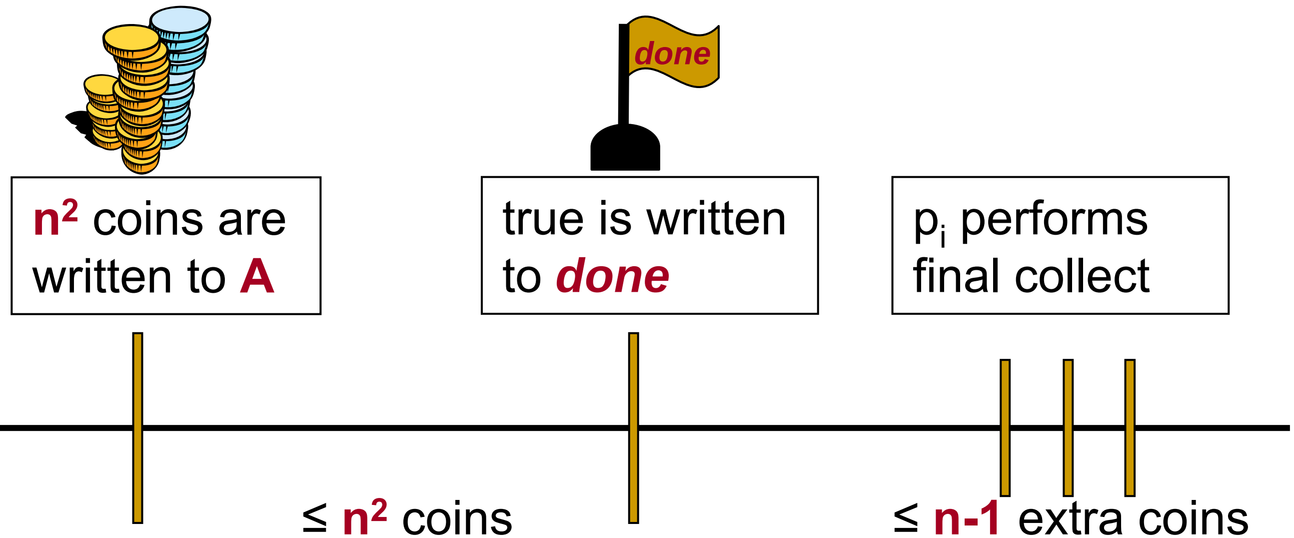
Shared Coin: Raising the Flag



Shared Coin Algorithm

```
while not done do
   A[i] = {count++, sum+random(-1,+1)}
   every n coin flips
       collect A
       if > n² coins were flipped
             then done = true
collect A
return the majority value of the coin flips
```

Total Step Complexity



Writing $O(n^2)$ coins, and reading *done* ⇨ $O(n^2)$ operations,
Every n coins a process performs a collect (n read operations) ⇨ $O(n^2)$ operations

Agreement Parameter: Overview

n^2 coins are written to **A**

true is written to ***done***

p_i performs final collect

≤ **n^2** coins

≤ **n-1** extra coins

sum ≥ **3n** with constant prob.

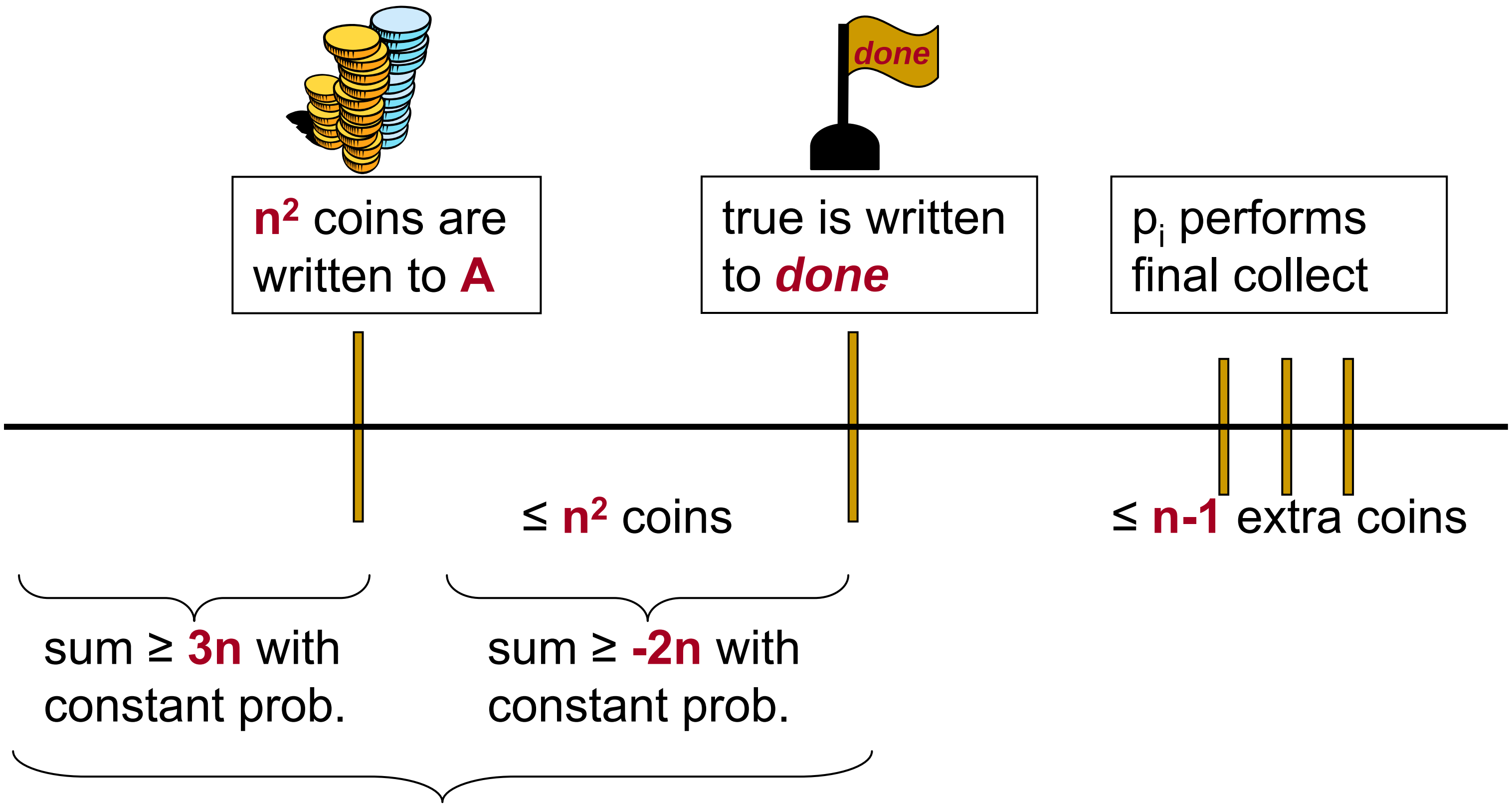
sum ≥ **-2n** with constant prob.

Central limit theorem

$$\Pr\left[sum \ge x\sqrt{N} \right] \xrightarrow{n\to\infty} 1-\Phi(x)$$

The stopping time is under the control of the adversary. Partial sums of a stochastic process (Kolmogorov ineq.)

Agreement Parameter: Summing Up



⇨ sum ≥ **n** with constant prob.

⇨ extra coins cannot make sum negative

⇨ every output is **1** with constant prob. (same for **-1**)

Partial Sums of Random Variables

A sequence of independent random variables $X_1, X_2, \ldots, X_m$ with $E[X_i]=0$ and $Var[X_i]=1$

Yields a sequence of partial sums $S_j = X_1+\ldots+X_j$

$E[S_j] = E[X_1+\ldots+X_j] = E[X_1]+\ldots+E[X_j] = 0$

$Var[S_j] = Var[X_1+\ldots+X_j] = Var[X_1]+\ldots+Var[X_j] = j$

Kolmogorov's Inequality: $\Pr\left[\max_{1\le j\le m} |S_j| \ge \lambda\right] \le \frac{1}{\lambda^2} Var\left[S_m\right]$

$$m = n^2, \lambda = 2n: \quad \Pr\left[\max_{1\le j\le n^2} |S_j| \ge 2n\right] \le \frac{1}{4n^2} n^2 = \frac{1}{4}$$

⇨ $S_j \ge -2n$ for every $1 \le j \le n^2$, with prob. 3/4

So…

- The total step complexity of randomized consensus can be improved.

… but only so much

Lower Bounds: A Brief History

History repeats itself. ~~Historians~~ computer scientists repeat each other

- $\Omega(n^2/\log^2 n)$ coin flips (asynchronous sm)

  [Aspnes 1998]

- $\Omega(\sqrt{(n/\log n)})$ rounds (synchronous mp)

  [Bar-Joseph & Ben-Or 1998]

⇨ $\Omega(n\sqrt{(n/\log n)})$ total step complexity (asynchronous mp/sm)

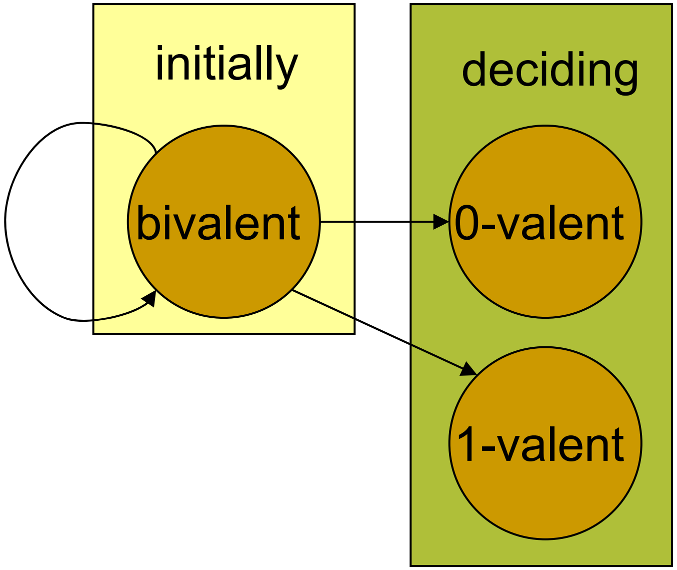
  ⇨ Worse than previous bound

FLP: Valency

0-valent: only deciding 0

1-valent: only deciding 1

bivalent: deciding 0
and deciding 1



Valency w/ Randomization

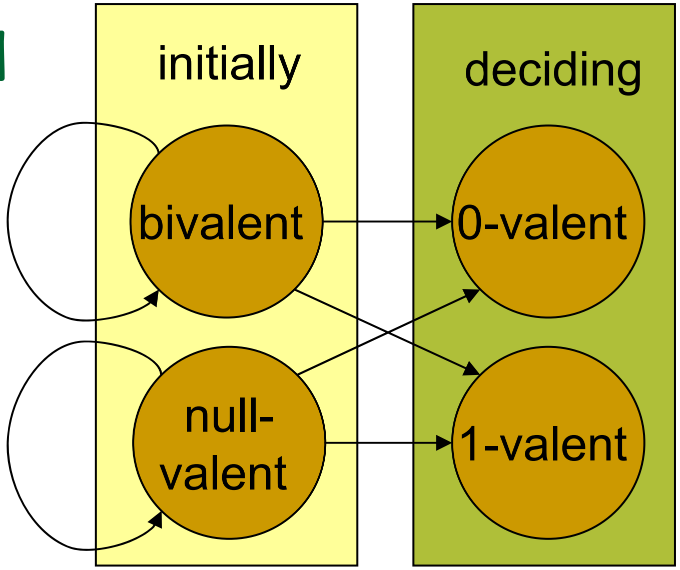
[Aspnes 1998]

0-valent: high probability for deciding 0

1-valent: high probability for deciding 1

bivalent: high probability for deciding 0
and high probability for deciding 1

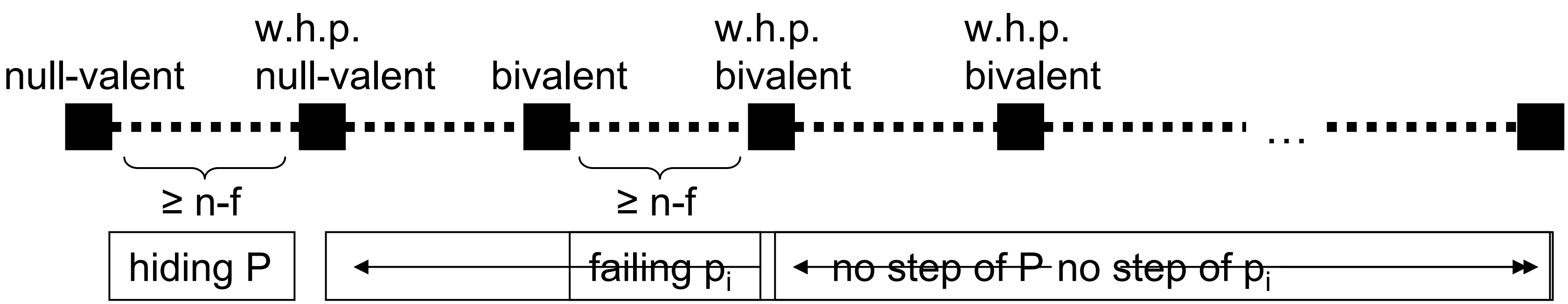
null-valent: **not** high probability for deciding 0
and **not** high probability for deciding 1



The Synchronous Lower Bound

[Bar-Joseph & Ben-Or 1998]

Executions proceed in rounds

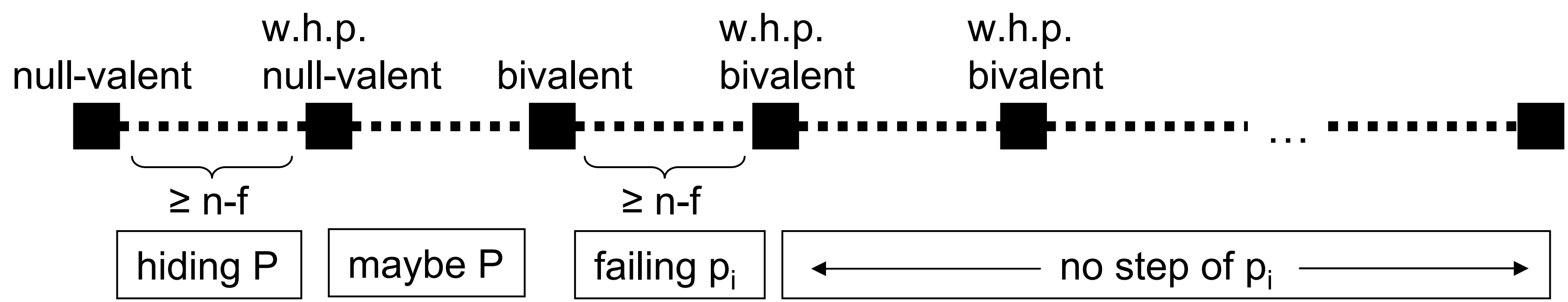


May need to hide $O(\log n \sqrt{n})$ processes in a round
⇨ cannot go for more than $\Omega(\sqrt{(n/\log n)})$ rounds

- An asynchronous process can be delayed without failing
  - Synchronous model w/ mobile failures
    ⇔ asynchronous model [Santoro, Widmayer 1989]

Our Asynchronous Lower Bound

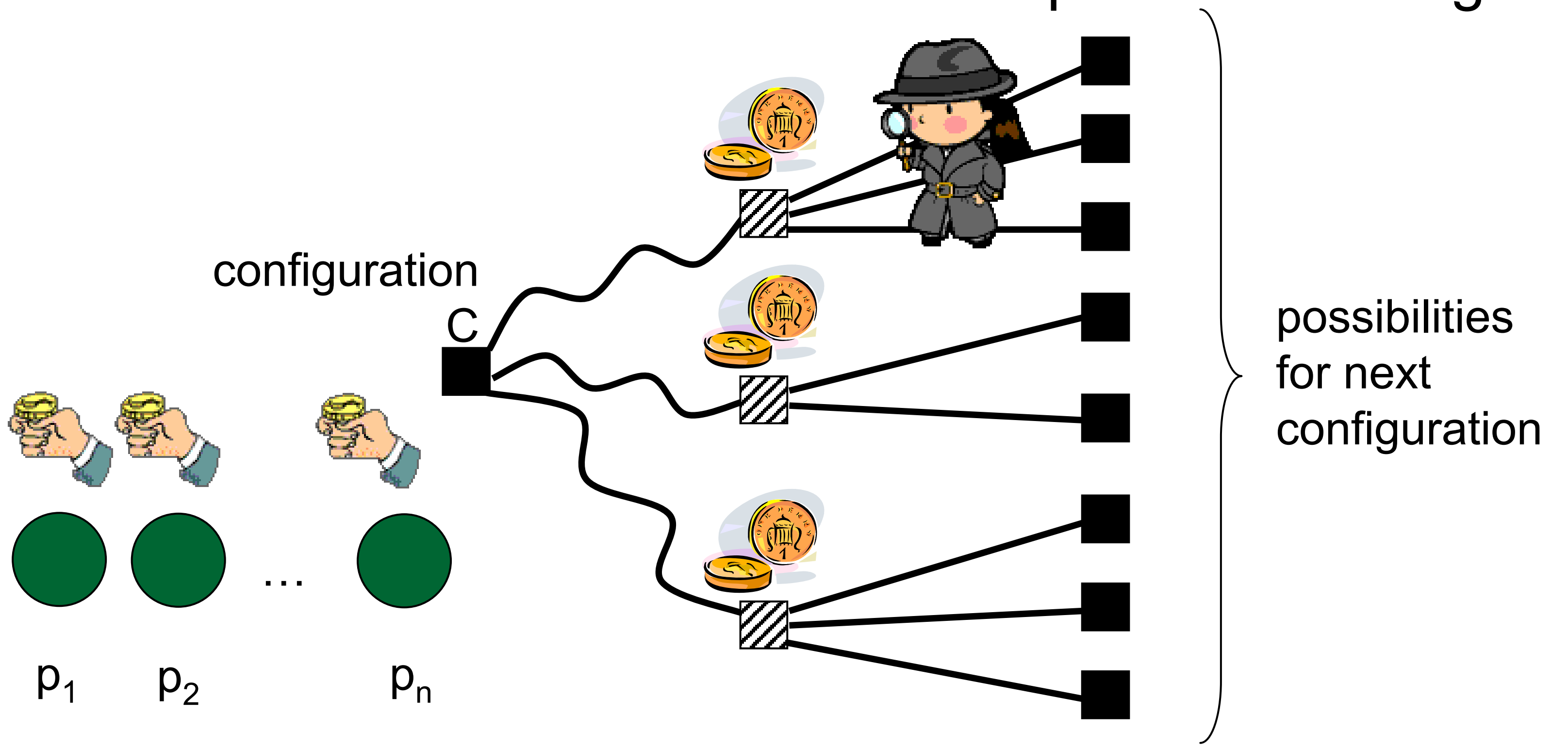
- Executions proceed in layers (sequence of ≥ n-f distinct processes) [Moses & Rajsbaum 1998]



- An asynchronous process can be hidden, by delaying, without failing
- f layers, each with at least n-f steps ⇨ f(n-f) steps

Strong Adversary, In More Detail

Two sources of non-determinism: coin-flips & scheduling



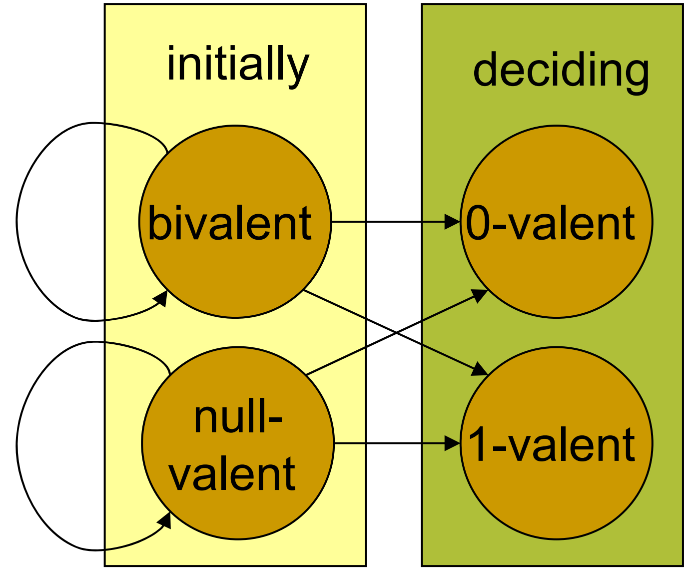
Valency, In More Detail

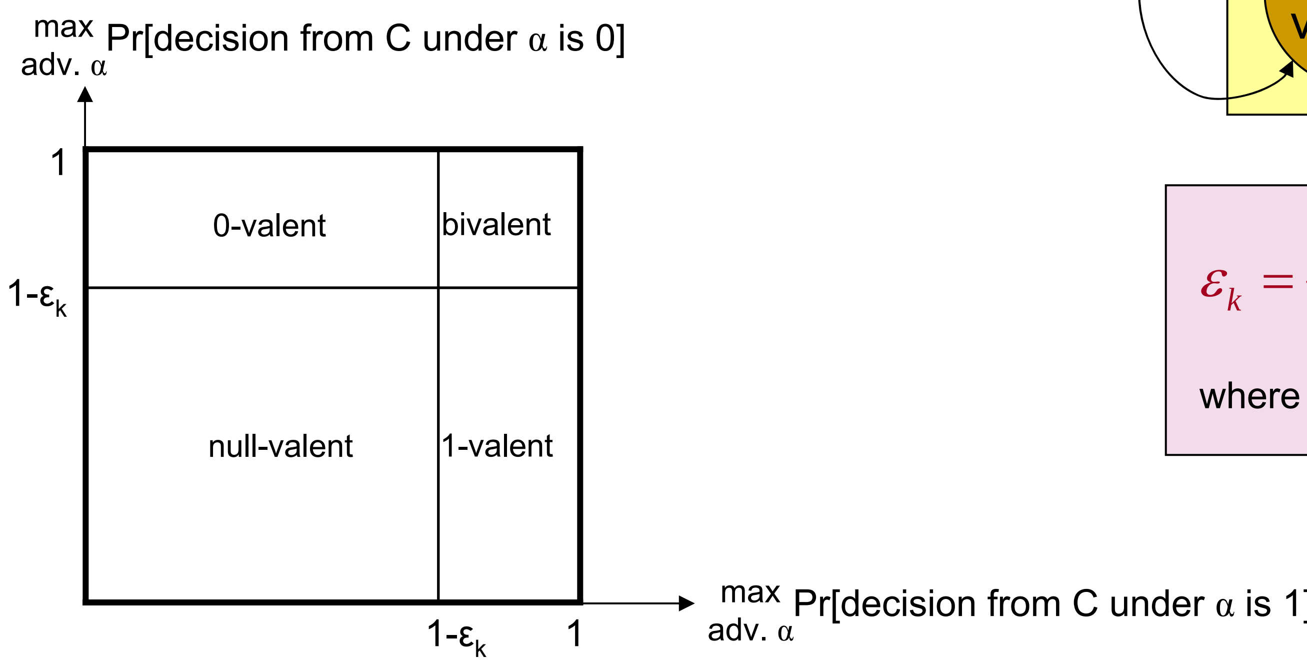
0-valent: high probability for deciding 0

1-valent: high probability for deciding 1

bivalent: high probability for deciding 0 and high probability for deciding 1

null-valent: **not** high probability for deciding 0 and **not** high probability for deciding 1





$$\varepsilon_k = \frac{1}{n\sqrt{n}} - \frac{k}{(n-f)^3}$$

where **k** is the layer number

Remaining Null-Valent

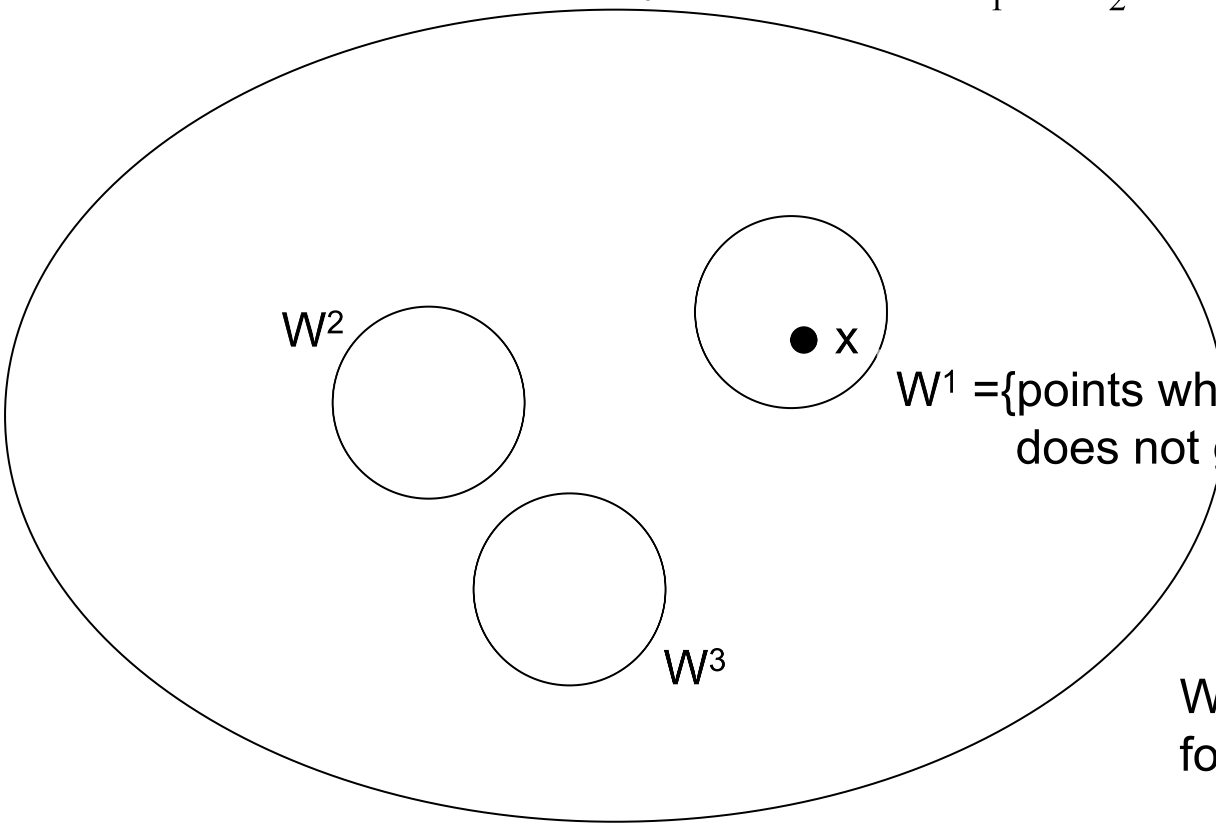
$$g:\{X_1 \cup \perp\} \times \{X_2 \cup \perp\} \times \ldots \times \{X_n \cup \perp\} \to \{1,2,3\}$$

3-valued one-round coin-flipping game, X_i is a random variable

Theorem: one outcome has high probability

One Outcome has High Probability

The probability space $X = X_1 \times X_2 \times \ldots \times X_n$



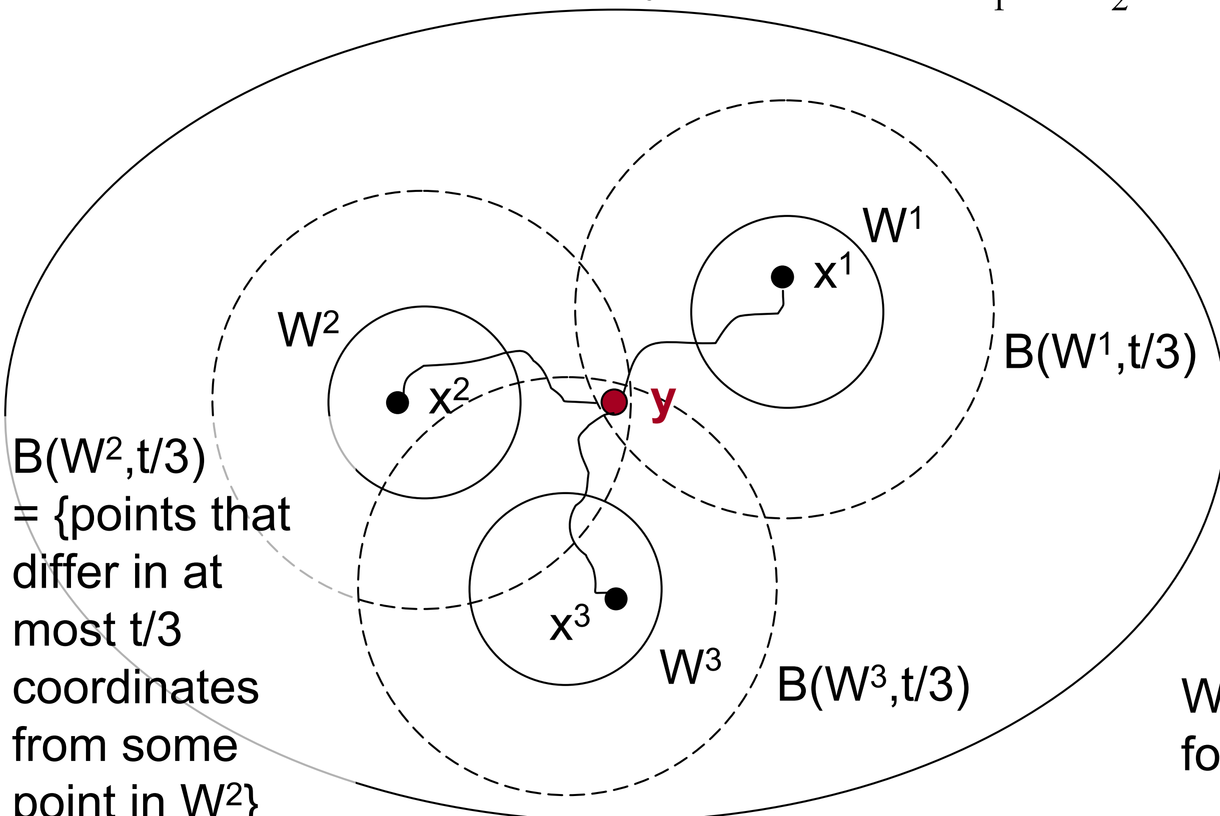
W^1 ={points where hiding ≤t coordinates does not give outcome 1}

We want to prove that for some u, **Pr[W^u]<1/n^3**

x=	x_1				…	…				x_n	g(x)≠1
$\|I\| \le t$ x_I=	x_1			$\perp$			$\perp$	$\perp$		x_n	$g(x_I) \neq 1$

One Outcome has High Probability

The probability space $X = X_1 \times X_2 \times \ldots \times X_n$



B(W²,t/3) = {points that differ in at most t/3 coordinates from some point in W²}

$$\overset{\neq 1}{g(y_I)=}\;\overset{\neq 2}{g(x^1_I)=}\;\overset{\neq 3}{g(x^2_I)=}\;g(x^3_I)=?$$

$y_I=$	y_1	$\perp$	…	$\perp$	$\perp$	y_n
$x^1_I=$		$\perp$		$\perp$	$\perp$	
$x^2_I=$		$\perp$		$\perp$	$\perp$	
$x^3_I=$		$\perp$		$\perp$	$\perp$	

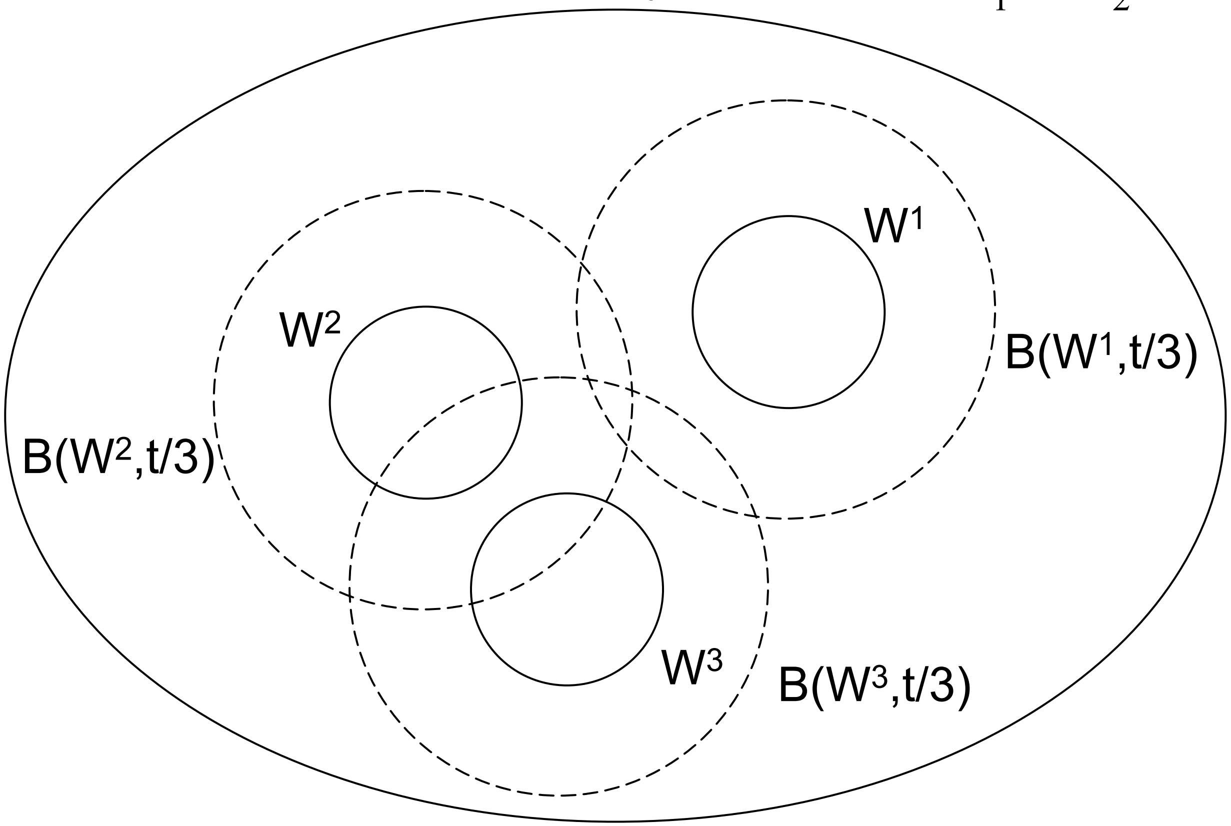
We want to prove that for some u, **Pr[W^u]<1/n^3**

W^u – adversary cannot reach outcome u. Assume **Pr[W^u]≥1/n^3**

Isoperimetric inequality: if **Pr[W^u]≥1/n^3** then **Pr[B(W^u,t/3)]≥1-1/n^3**

One Outcome has High Probability

The probability space $X = X_1 \times X_2 \times \ldots \times X_n$



W^u – adversary cannot reach outcome u.

For some u, **$Pr[W^u]<1/n^3$**.

The adversary can reach the outcome u with probability **$\geq 1-1/n^3$**.

Must be the null-valent category.

Remaining Null-Valent

$$g : \{X_1 \cup \perp\} \times \{X_2 \cup \perp\} \times \ldots \times \{X_n \cup \perp\} \rightarrow \{1, 2, 3\}$$

Product probability space = results of local coin-flips

$\perp$ stands for a process not taking a step in the layer

- 0-valent or bivalent configuration → the outcome of **g** is **1**
- 1-valent configuration → the outcome of **g** is **2**
- Null-valent configuration → the outcome of **g** is **3**

⇨ some outcome can be forced (by hiding processes) w.h.p.

Since we started from a null-valent configuration

1. not high probability of deciding 0
   ⇨ cannot have high probability for reaching a 0-valent configuration
2. not high probability of deciding 1
   ⇨ cannot have high probability for reaching a 1-valent configuration

⇨ the **null-valent** category must be the one with high probability

Must Remain Null-Valent

- Null-valent configuration has probability for deciding 0 at most 1-ε_k
- Assume 0-valent or bivalent configuration can be reached with probability 1-1/m^3
- New configuration has probability for deciding 0 at least 1-ε_{k+1}
- Together, the probability for deciding 0 from the null-valent configuration is:

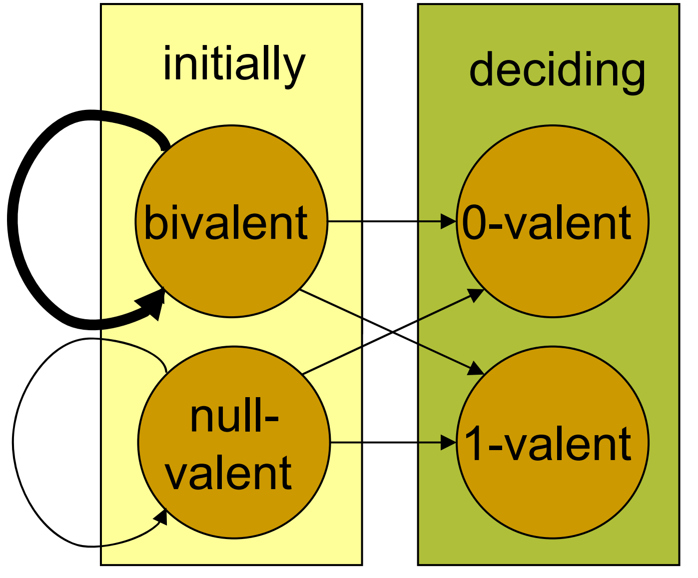
$$\geq \left(1-\frac{1}{(n-f)^3}\right)\left(1-\varepsilon_{k+1}\right)=\left(1-\frac{1}{(n-f)^3}\right)\left(1-\frac{1}{n\sqrt{n}}+\frac{k+1}{(n-f)^3}\right)$$

$$=1-\frac{1}{n\sqrt{n}}+\frac{k}{(n-f)^3}+\frac{1}{(n-f)^3 n\sqrt{n}}-\frac{k+1}{(n-f)^6}$$

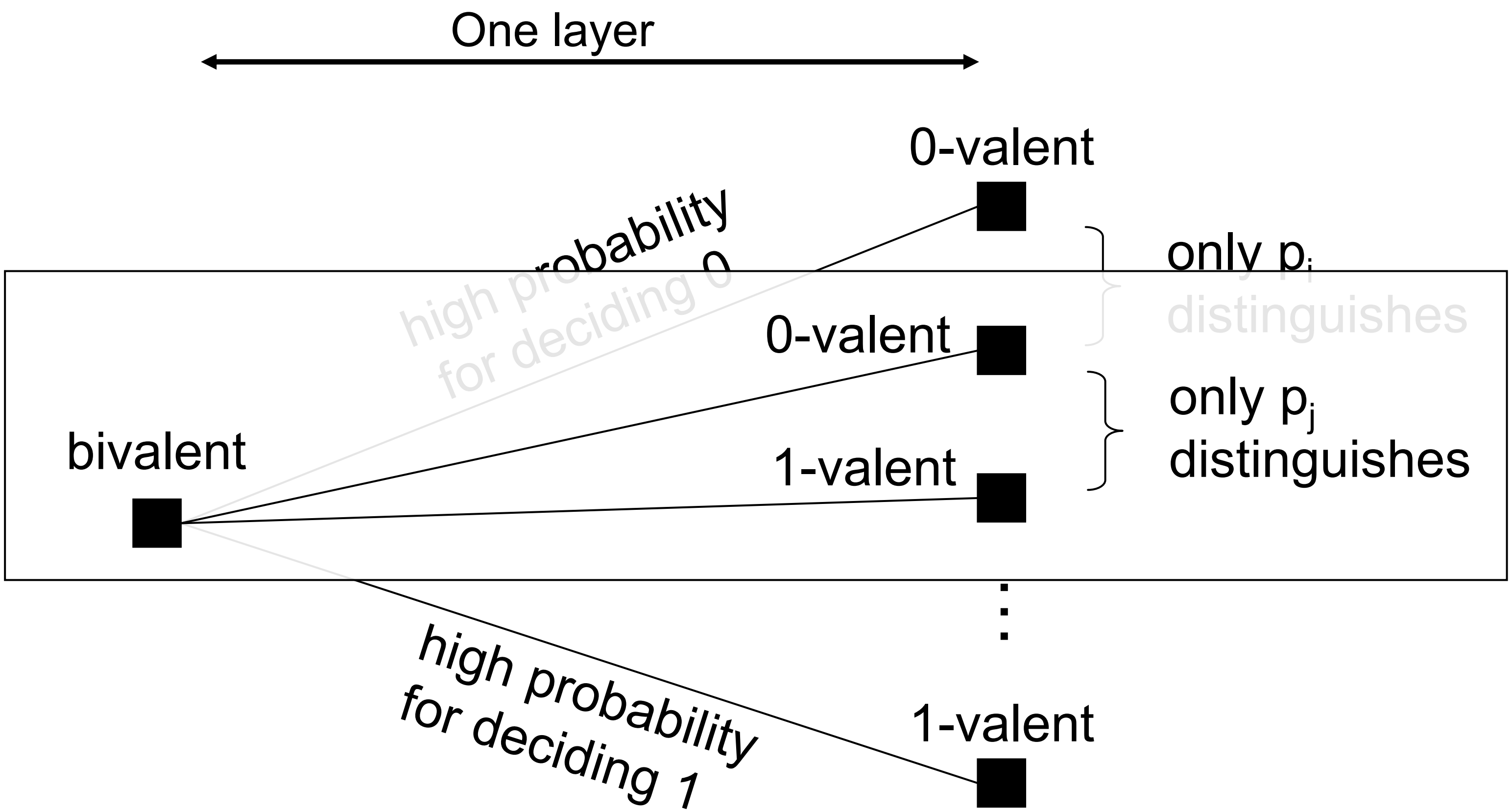
$$>1-\frac{1}{n\sqrt{n}}+\frac{k}{(n-f)^3}=1-\varepsilon_k$$

Cannot have high probability for reaching a 0-valent or bivalent configuration

Big Picture: Bivalent Configurations

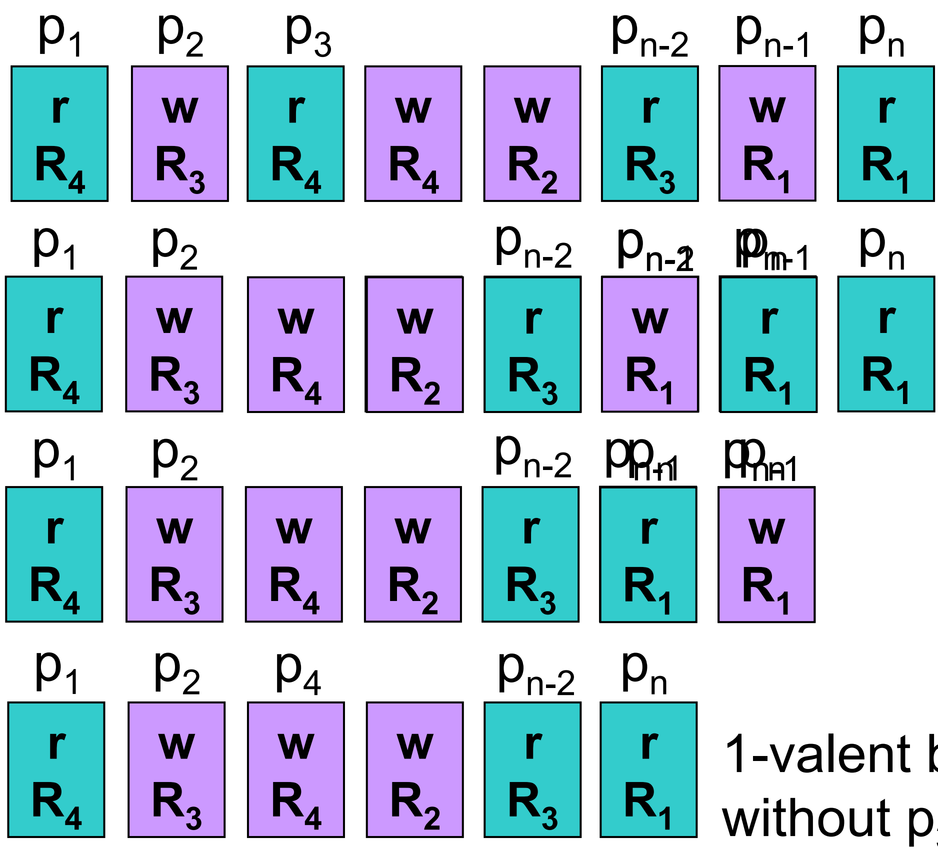


Bivalent Configurations: Connectivity



fail p_i ⇨ must have same valence ⇨ null-valent

Connectivity in Shared Memory



0-valent by a full layer

only p_3 distinguishes

both p_{n-1} and p_n distinguish

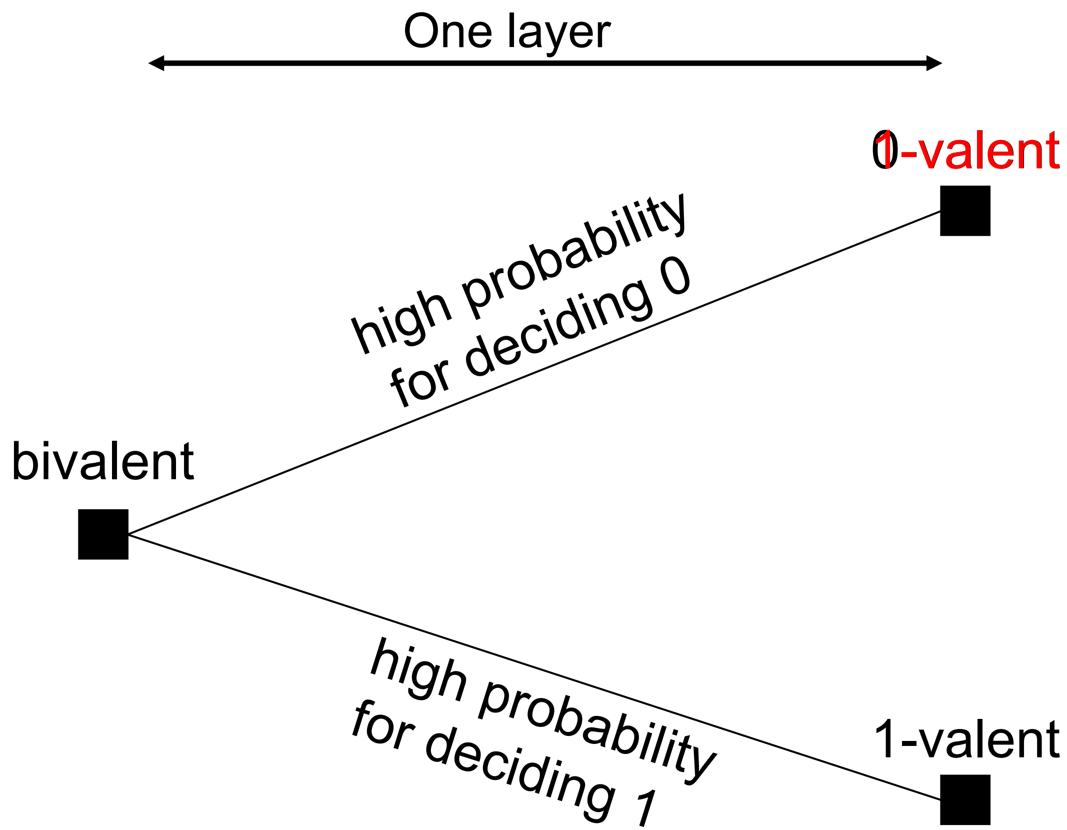
only p_n distinguishes

1-valent by a layer without p_3 and p_{n-1}

only p_{n-1} distinguishes

0-valent and 1-valent configurations which are distinguishable only to p_i

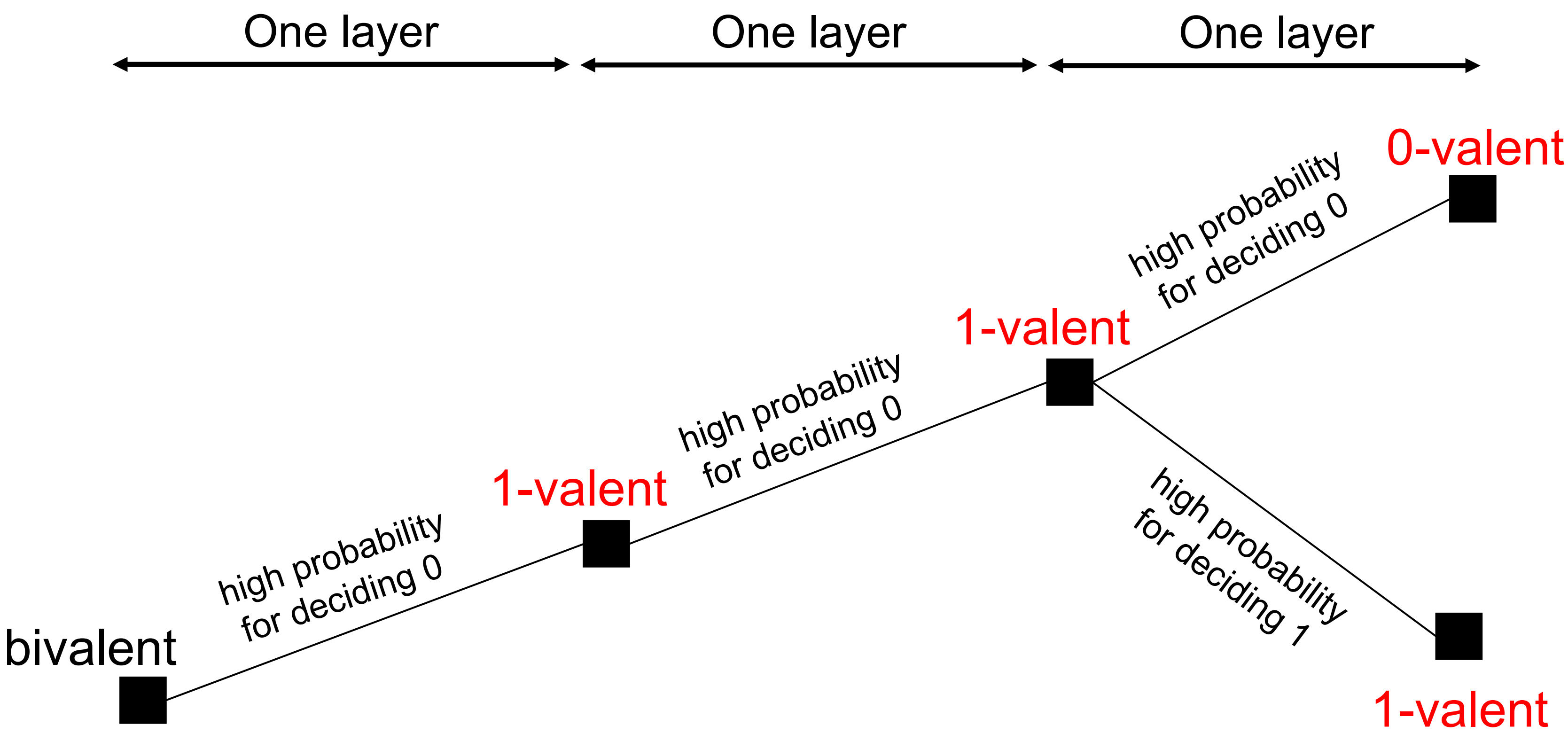
Bivalent Configurations Revisited



~~Adversary with high probability for deciding 0 → 0-valent configuration~~

Cannot even guarantee it with high probability

v-Switching Configurations



The above 0-valent configuration exists with high probability (at least $1-\varepsilon_k$)

And now continue as from a bivalent configuration

Individual Step Complexity

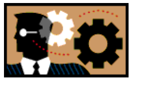
The expected number of steps taken by a single process

- **$O(n\log^2 n)$** using single-writer registers [Aspnes & Waarts 1996]
- In our algorithm, a process running solo must generate all **n^2** coins alone

A multi-writer register allows randomized consensus with **$O(n\log n)$** individual step complexity

- The coins of a process have increasing weights
- A process that runs alone flips only **$O(n\log n)$** coins
- But now the coins are not independent – weights of flipped coins can be different

Wrap-Up: What's this Work About?

- At face value: Clever math…
  - Stochastic processes, martingales, Kolmogorov
- Really, confluence of models
  - Asynchronous ⇔ synchronous w/ mobile failures
  - Taking connectivity arguments from message passing to (multi-writer) shared memory
  - Multi-writer bit induces instantaneous views
- Still needs to do your math… 

What's Next?

Randomized consensus revisited…

- ✓ Weaker adversaries?
- Single-writer registers?
- Message-passing model?
- Byzantine failures?
  - Cryptographic requirements for BFT?

And other problems…

- k-set consensus, renaming
- Seems to defy existing techniques