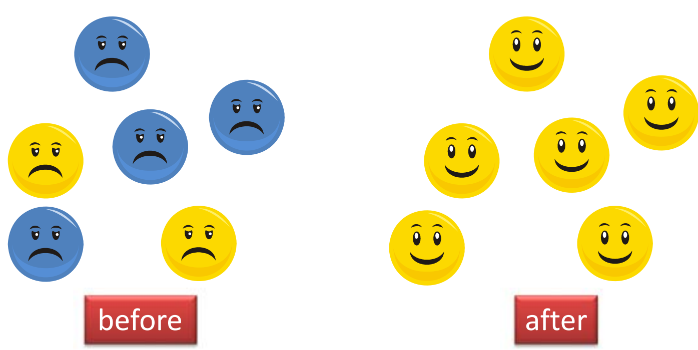
236755
Topic 3: Consensus

Winter 2019-20

Prof. Hagit Attiya

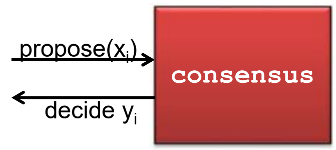
Consensus

Processes propose values and must agree on a common decision value, proposed by some process



Consensus, More Formally

Each process p_i proposes a binary **input value**, x_i, and returns an **output value** (**decision**), y_i



Only two conditions on the output:

Agreement: all processes decide on the same value

Validity: this value is the proposal of some process

Consensus is Easy…

If processes can wait for each other.

```
propose(v)
    R[i] = v
    while true
        read R[1,…,n] to l[1,…,n]
        if all l[1,…,n] ≠ ⊥
            decide on min l[1,…,n]
            (and return)
```

Termination (Liveness)

Solo-termination (also called **obstruction-freedom**):
A process has to terminate if (eventually) it runs by itself.

Termination (Liveness)

Solo-termination (also called **obstruction-freedom**):
A process has to terminate if (eventually) it runs by itself.

Nonblocking: From any configuration, if some process takes infinitely many steps, then some process (not necessarily the same one) terminates, regardless of steps by other processes.
(Analogue of no deadlock for mutex.)

Termination (Liveness)

Solo-termination (also called **obstruction-freedom**):
A process has to terminate if (eventually) it runs by itself.

Nonblocking: From any configuration, if some process takes infinitely many steps, then some process (not necessarily the same one) terminates, regardless of steps by other processes.

lock-free

Wait-freedom: From any configuration, if some process takes infinitely many steps, then the process terminates, regardless of steps by other processes.
(Analogue of no starvation for mutex.)

Consensus with Compare&Swap

Wait-free consensus for any number of processes using compare&swap is easy

```
compare&swap(R,old,new):
   temp = R
   if R == old
      R = new
   return temp
```

Use a single shared variable, **`first`**, initially ⊥

```
propose(x)
    v = cas(first,⊥,x)
    if v == ⊥ decide x
    else decide v
```

What happens if we use only reads and writes?

Graded Consensus (Adopt-Commit)

Like consensus but the decision is **(grade, y_i)**, grade is either **adopt** or **commit**, such that

Graded agreement: if a process decides **(commit,y_i)** then all processes decide either **(adopt,y_i)** or **(commit,y_i)**

Validity: **y_i** was proposed by some process

Convergence: If only **y_i** is proposed before p outputs **(grade,y_i)** then **grade = commit**

- Two special cases: (a) p runs alone (b) same proposals

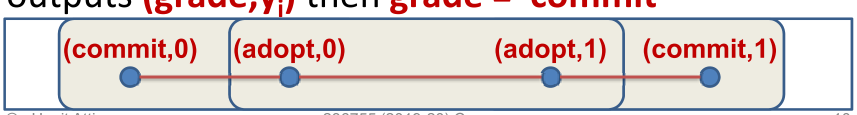
Graded Consensus (Adopt-Commit)

Like consensus but the decision is **(grade, y_i)**, grade is either **adopt** or **commit**, such that

Graded agreement: if a process decides **(commit,y_i)** then all processes decide either **(adopt,y_i)** or **(commit,y_i)**

Validity: **y_i** was proposed by some process

Convergence: If only **y_i** is proposed before p outputs **(grade,y_i)** then **grade = commit**



Wait-Free Adopt-Commit Provides Solo Terminating Consensus

Use infinitely many copies of adopt-commit$_1$, …

```
propose(v)
  for ( m = 1; ; m++)
      (grade,value) = adopt-commit_m(v)
      if grade == commit return value
      else v = value
```

Validity and agreement are immediate from the related properties of the adopt-commit protocol

Solo termination follow from convergence (a)

Adopt-Commit with SW Registers

Two arrays of single-writer variables A[1,…,n] B[1,…,n], all initially ⊥

must be v

```
adopt-commit(v)
    write v to A[i]
    read A[1],…,A[n]              // one by one
    if only one non-⊥ value w, B[i] = "commit w"
    else B[i] = "adopt v"
    read B[1],…,B[n]              // one by one
    if only "commit w", return (commit,w)
    else if contains "commit w", return (adopt,w)
    else return (adopt,v)
```

Proof of SW Adopt-Commit

✓ Validity is clear
✓ Convergence follows from inspecting the code

```
adopt-commit(v)
    write v to A[i]
    read A[1],…,A[n]          // one by one
    if only one non-⊥ value w, B[i] = "commit w"
    else B[i] = "adopt v"
    read B[1],…,B[n]          // one by one
    if only "commit w", return (commit,w)
    else if contains "commit w", return (adopt,w)
    else return (adopt,v)
```


Graded Agreement of SW Adopt-Commit

Lemma: If p_i writes "commit v" to B[i], then no process writes "commit w" to B, w ≠ v

v must be the first value written in the array A

If p_i returns (commit,v), then "commit v" is the first value written in B
⇨ all processes commit or adopt v

```
adopt-commit(v)
    write v to A[i]
    read A[1],…,A[n]          // one by one
    if only one non-⊥ value w, B[i] = "commit w"
    else B[i] = "adopt v"
    read B[1],…,B[n]          // one by one
    if only "commit w", return (commit,w)
    else if contains "commit w", return (adopt,w)
    else return (adopt,v)
```

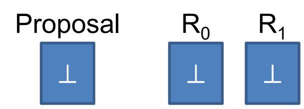
Adopt-Commit with MW Registers

shared variables:

Proposal, initially ⊥

R_0, R_1, initially ⊥

local variable preference

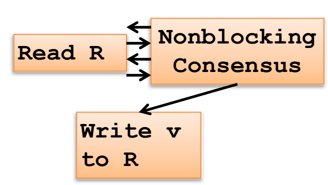



```
adopt-commit(v)
    R_v  =  1
    if Proposal ≠ ⊥ preference = Proposal
    else
      preference = v
      Proposal = preference
    if R_{1-v} ≠ ⊥ return (adopt, preference)
    else return (commit, preference)
```


What about Wait-Free Consensus?

Nonblocking consensus implies wait-free consensus (using a mw register R)

- Execute the nonblocking consensus
- After deciding, write decision to R
- Interleave reading R with nonblocking consensus



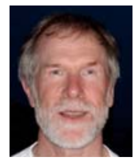
What about Wait-Free Consensus?

Wait-free consensus is impossible
Even if we can wait for all but one process

Fischer

Lynch

Patterson

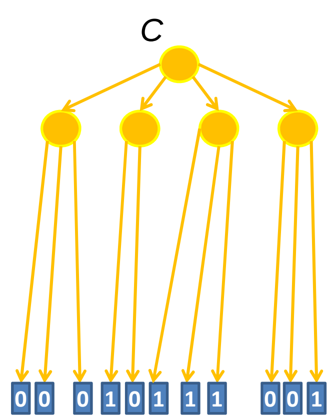


Implies the same for nonblocking consensus

Holds for message passing and shared memory

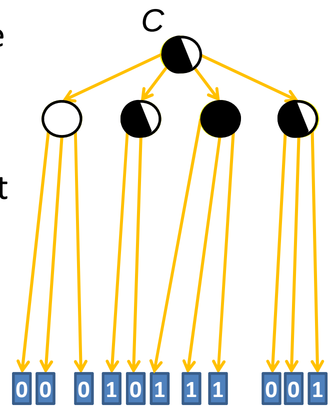
Potence of a Configuration C

- *C* is ***v*-potent** if v is decided in some configuration reachable from *C*



Valence of a Configuration C

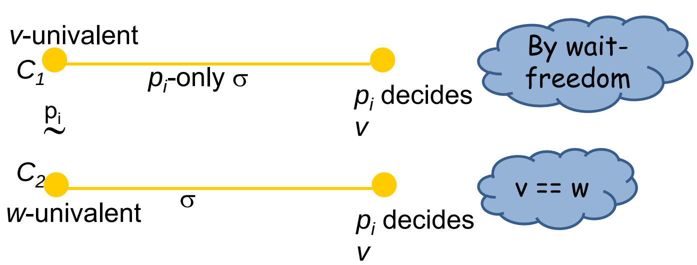
- *C* is ***v*-potent** if v is decided in some configuration reachable from *C*
- *C* is ***v*-univalent** if it is v-potent but not *(1-v)*-potent (**univalent** in general)
- *C* is **bivalent** if it is both 0-potent and 1-potent



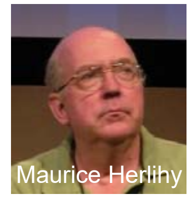
Univalent Similarity

Lemma: If C_1 and C_2 are both univalent and they are *similar* w.r.t. some process p_i, then they have the same valence.

Proof:



Impossibility of Two-Process Consensus using Reads / Writes

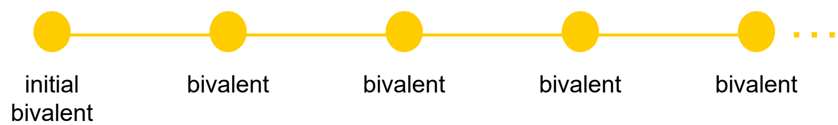


Proof overview: If there is a 2-process wait-free consensus algorithm

- Show there is a bivalent initial configuration
- Show that from every bivalent configuration there is an execution leading to a bivalent configuration

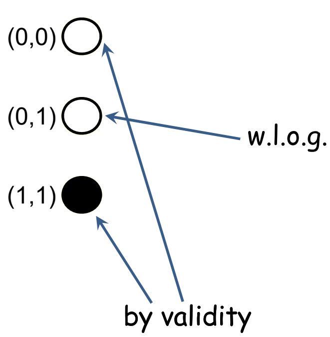
⇨No process decides

For simplicity assume single-writer variables



There is a Bivalent Initial Configuration

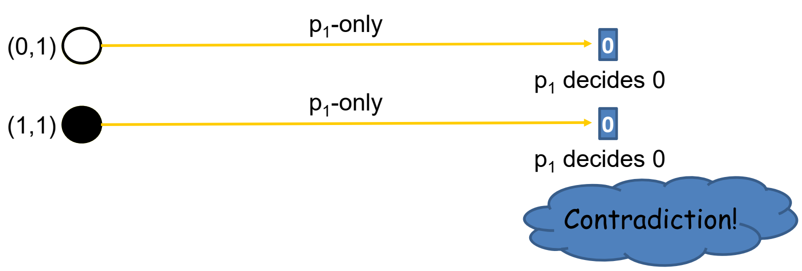
Assume all initial configurations are univalent



There is a Bivalent Initial Configuration

Assume all initial configurations are univalent

These configurations look the same to p_1



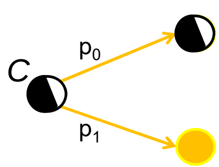
Extending a Bivalent Configuration

Lemma: If C is bivalent then a bivalent configuration C' is reachable from C.

Consider the configurations reachable by a single step of each process

If either of them is bivalent ⇨ we are done

Both are univalent ⇨ 1-valent & 0-valent



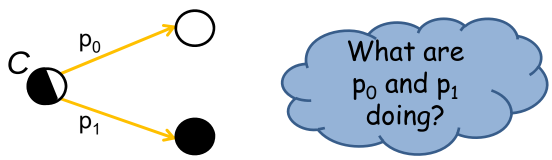
Extending a Bivalent Configuration, One Step at a Time

Lemma: If C is bivalent then a bivalent configuration C' is reachable from C.

Consider the configurations reachable by a single step of each process

If either of them is bivalent ⇨ we are done

Both are univalent ⇨ 1-valent & 0-valent

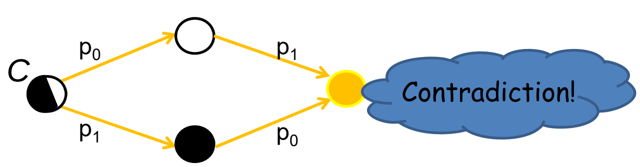


Extending a Bivalent Configuration, Steps that Commute

Lemma: If C is bivalent then a bivalent configuration C' is reachable from C.

Case 1: both read or both write (different variables)

Their steps commute

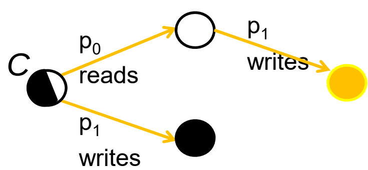


Extending a Bivalent Configuration, Overwriting Step

Lemma: If C is bivalent then a bivalent configuration C' is reachable from C.

Case 2: p_0 reads, p_1 writes (or vice versa) to the same variable

Covering…

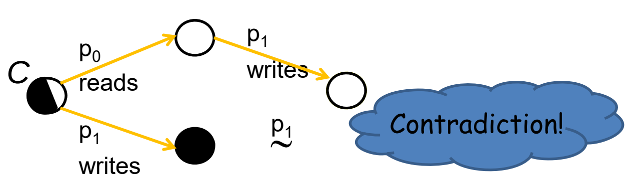


Extending a Bivalent Configuration, Overwriting Step

Lemma: If C is bivalent then a bivalent configuration C' is reachable from C.

Case 2: p_0 reads, p_1 writes (or vice versa) to the same variable

Look the same to p_1



The Full Impossibility Result

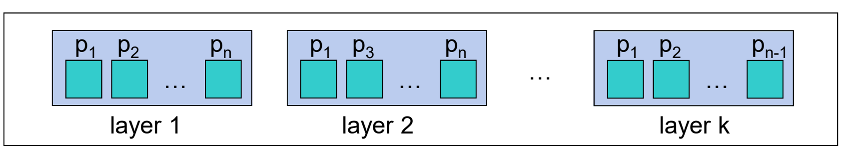
- In an asynchronous system, consensus cannot be solved when the algorithm has to tolerate **even just a single failure**
  - n-1 processes cannot take an infinite number of steps without deciding
- Holds for the shared-memory model **as well as for the message-passing** model
- Describe both proofs in a unified manner using **layered executions**
  - below, f is the number of processes that may fail, we concentrate on the case f = 1

Layered Schedules

f-layer: sequence of at least n-f different processes

- Order is sometimes important

f-schedule: sequence of f-layers



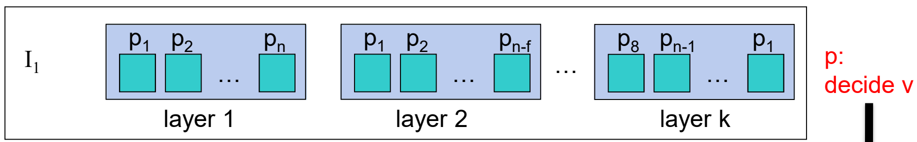
- p_2 is **faulty** in layer 2, but nonfaulty in layer k
- p_n **crashes** in layer 3 – faulty in every layer r, $3 \le r \le k$

An f-schedule σ and an initial configuration I determine a **layered execution** α(σ, I)

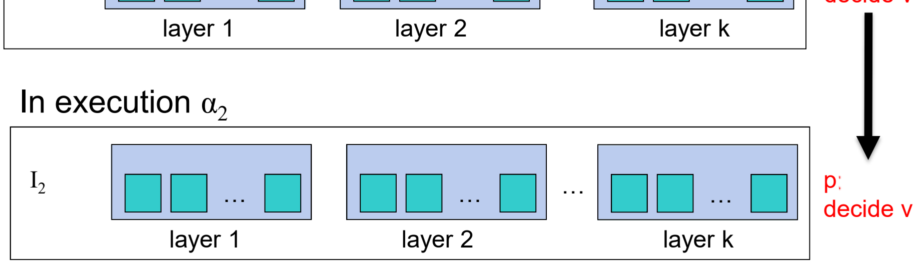
Similarity

$\alpha_1 \overset{p}{\sim} \alpha_2$

In execution α_1

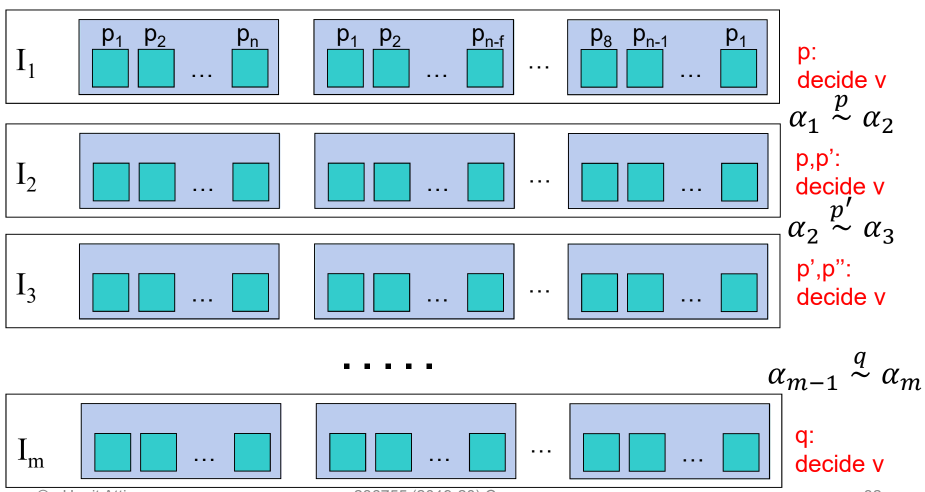


In execution α_2



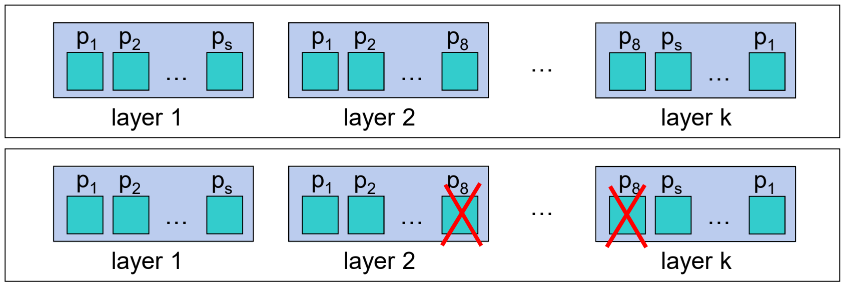
Connectivity

$\alpha_1 \approx \alpha_m \equiv \alpha_1 \overset{p}{\sim} \alpha_2 \overset{p'}{\sim} \alpha_3 \sim \ldots \overset{q}{\sim} \alpha_m \Rightarrow$ same decision



Key Lemma: Crashing a Process

crash(σ,p,r): p crashes in layer r of σ



crash(σ,p_8,2)

Lemma: For every input configuration I, f-schedule σ, process p and round r, $\alpha(\sigma, I) \approx \alpha(crash(\sigma, p, \mathrm{r}), I)$

Deriving the Impossibility Result: Input Connectivity

Consider a sequence of input configurations

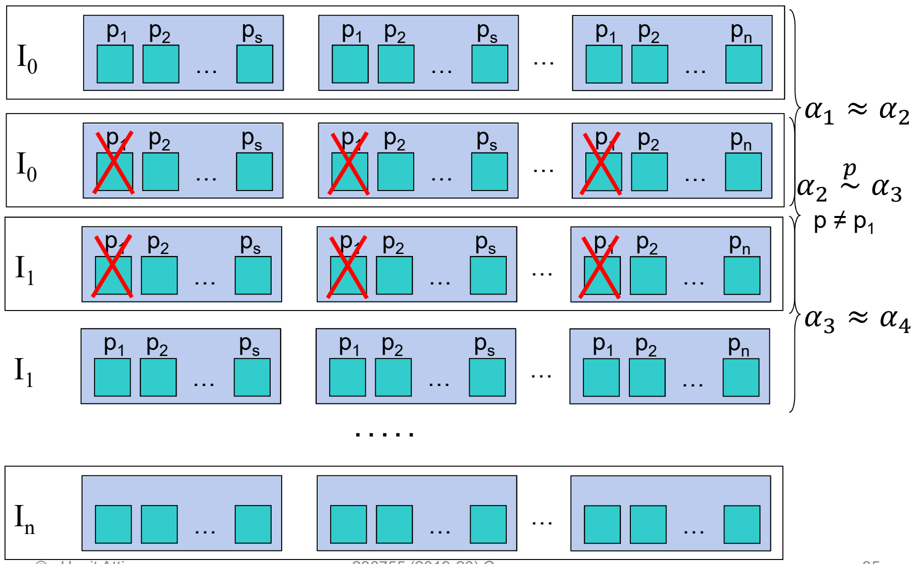
I_0 = (0,0,…,0)
$\approx$
I_1 = (1,0,…,0)
$\approx$
I_2 = (1,1,…,0)
…
$\approx$
I_n = (1,1,…,1)

Claim: $\alpha(\sigma_F, I_0) \approx \alpha(\sigma_F, I_n)$, where σ_F is the **full** layered schedule

⇨ Same decision in I_0 (all zeroes) and I_n (all ones)

Contradiction!

Getting the Claim from Key Lemma



© Hagit Attiya 236755 (2019-20) Consensus 35

Proving the Key Lemma

Lemma: For every input configuration I, process p and round r, $\alpha(\sigma, I) \approx \alpha(crash(\sigma, p, \mathrm{r}), I)$

The proof is very model dependent

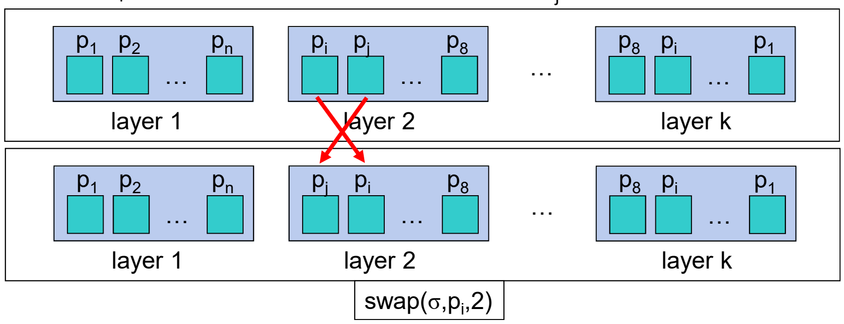
- Shared memory: read / write (single-writer) ✓
- Message passing

Need to assume bounded executions

© Hagit Attiya 236755 (2019-20) Consensus 36

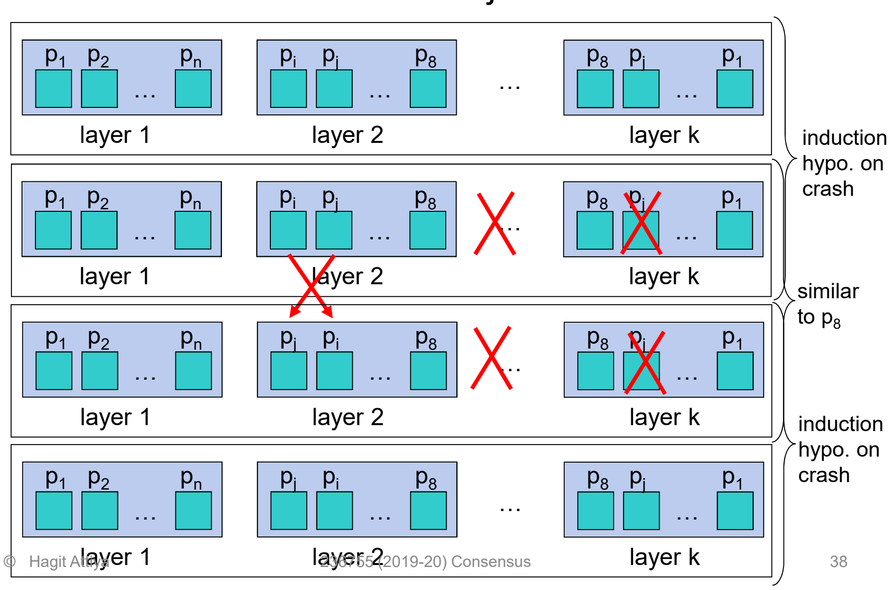
swap(σ,p_i,r)

Process p_i is swapped with the next process (p_j) in layer r



Both read or write (different registers) ⇨ no process distinguishes.
p_j reads and p_i writes ⇨ only p_j distinguishes at the end of layer r

swap(σ,p_i,r): only p_j distinguishes

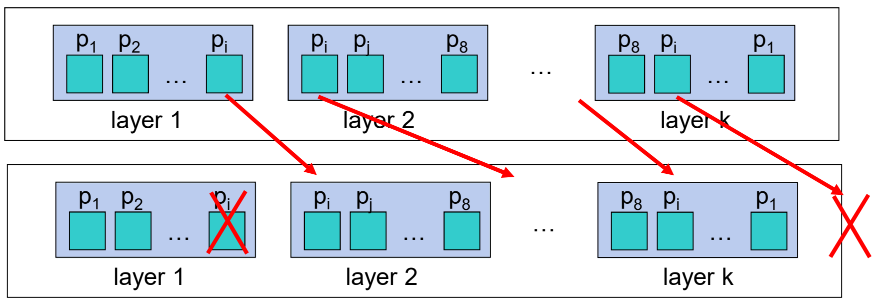


delay(σ,p_i,r)

The idea is to crash p_i by swapping every appearance of p_i until after the end of the schedule

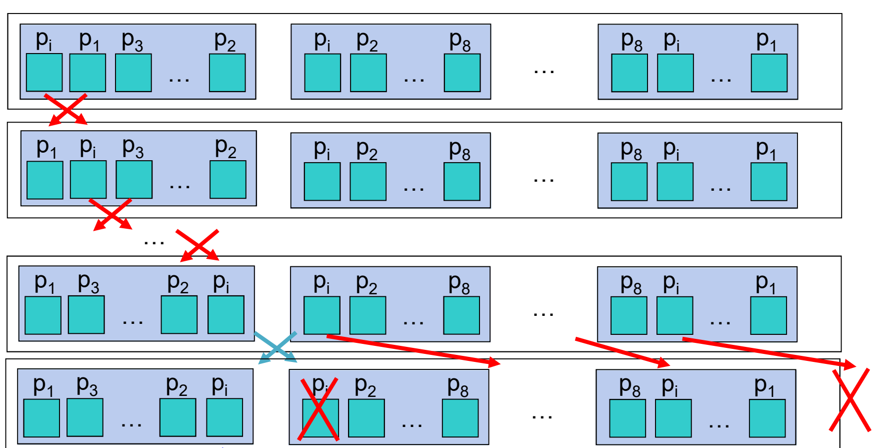
But cannot swap with another appearance of p_i

Delay p_i by one layer, starting from layer r



delay(σ,p_i,1)

Swaps + Delays ⇨ Crash



Now continue swapping…

Algebraically

$$\text{delay}(\sigma,p,r) = \text{swap}^{k}(\text{rollover}(\text{swap}^{k'}(\text{delay}(\sigma,p,r+1),p,r),p,r),p,r+1)$$

k' is p's distance from the end of layer r

k is p's distance from the beginning of layer r+1

Similarly

$$\text{crash}(\sigma,p,r) = \text{crash}(\text{delay}(\sigma,p,r),p,r+1)$$

Impossibility Result for Message-Passing Systems

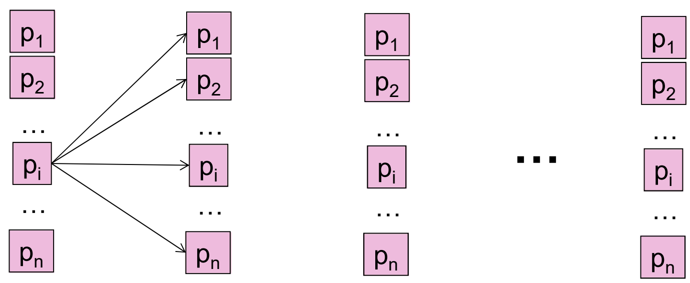
Original context of this result (FLP)

Original proof has a different structure (similar to previous lecture)

Message-passing: Model of Computation

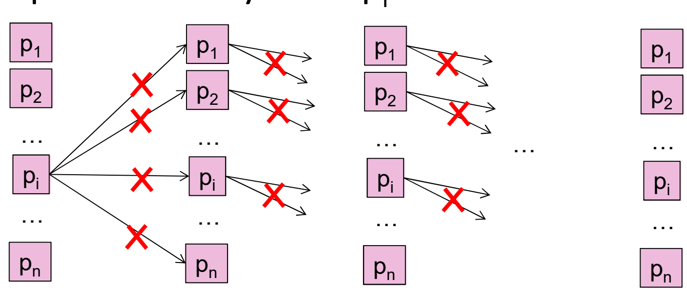
In each step, send messages to all processes

In layered executions, we synchronize the steps



Message-passing

- Crash p_i by removing it from all layers
  - Incremental ⇨ remove messages from p_i to p_j
  - Inductively, crash p_j in following layers
- Repeat for all layers ⇨ p_i crash



Bounding the Executions

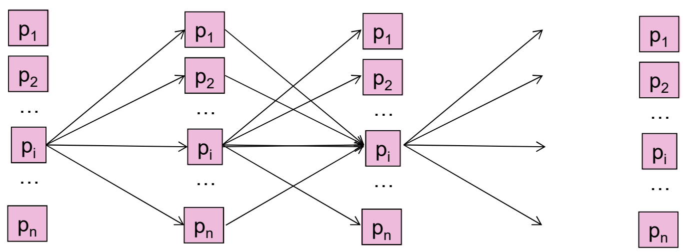
- Why?
- To have a well-defined base case for the (backwards) induction on the layer number
- How?
- The proof considers a fixed (and bounded) set of executions from n+1 input configurations

Consensus in Synchronous Systems

"Then we are agreed nine to one that we will say our previous vote was unanimous!"

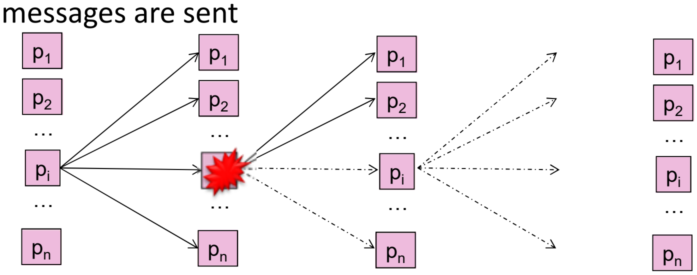
Synchronous Systems

- Processes take steps in **rounds**
- In each round, a process
  - sends messages to all (other) processes
  - receive messages from all other processes
  - does some local computation



Crash Failures in Synchronous Systems

- All but at most f **faulty processes** take an infinite number of steps (or until everyone decides)
- Once a faulty processor fails to take a step in a round, it takes no more steps
- In the last step of a faulty process, some subset of its outgoing messages are sent



Consensus Algorithm for Crash Failures

Ray Strong Danny Dolev

- Tolerates $f < n$ crash failures
- Requires $f + 1$ rounds

Consensus Algorithm for Crash Failures

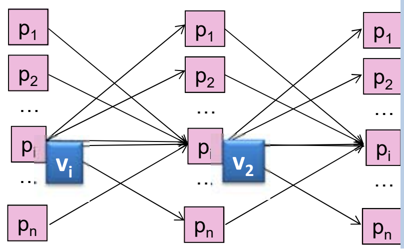
Each process executes the following code

```
v = my input
in each round 1 through f+1:
   if v not sent before, send v to all
   wait to receive messages for this round
   v = min of received values and current value of v
in round f+1, decide on v
```

- Tolerates $f < n$ crash failures
- Requires $f + 1$ rounds
- A total of $\leq n^2|V|$ messages
  each with $\log|V|$ bits, where V is the input set.

An Execution of the Algorithm: p_i with input v_i

```
v = my input
in each round 1 through f+1:
   if v not sent before, send v to all
   wait to receive messages for this round
   v = min of received values and current value of v
in round f+1, decide on v
```



- round 1:
  - send input
  - receive round 1 messages
  - compute value for v
- round 2:
  - send v (if this is a new value)
  - receive round 2 messages
  - compute value for v

Correctness of Crash Consensus Algorithm

Termination: By the code, finish in round *f+1.*

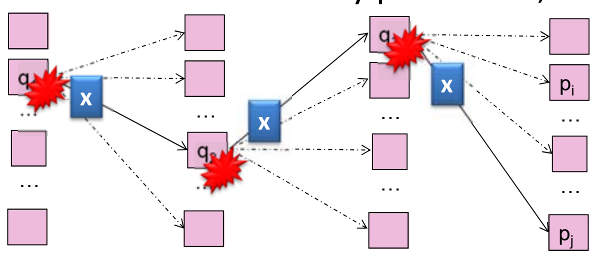
Validity: processes do not create values.
If all inputs are the same, then that is the only value ever sent around (and decided)

Crash Consensus Algorithm: Agreement

Suppose in contradiction p_j decides on a smaller value, x, than p_i does

⇨ x was hidden from p_i by a chain of faulty processes (one for each round)

⇨ This chain has f + 1 faulty processors, a contradiction



Is this the Best Round Complexity?

Rounds Lower Bound: Initial Lemma

Lemma: From some initial configuration, there are two executions γ and α, in which two different values are decided. γ is failure-free, and in α, one process crashes before taking any steps, but no other processes fail.

C_0 = (0,0,…,0,0) — v_0 = 0 (by validity)

C_i = (0,0,…,1,1) —failure-free— v_i is decided

C_n = (1,1,…,1,1) — v_n = 1 (by validity)

Rounds Lower Bound: Proof of Initial Lemma

Lemma: From some initial configuration, there are two executions γ and α, in which two different values are decided. γ is failure-free, and in α, one process crashes before taking any steps, but no other processes fail.

C_j = (0,0,…,0,1) —failure-free— 0 is decided

—p_j crashes at the start— 0 is decided?

C_{j+1} = (0,0,…,1,1) —failure-free— 1 is decided

✓

Rounds Lower Bound: Proof of Initial Lemma

Lemma: From some initial configuration, there are two executions γ and α, in which two different values are decided. γ is failure-free, and in α, one process crashes before taking any steps, but no other processes fail.

$C_j = (0,0,\ldots,0,1)$ — failure-free — 0 is decided

p_j crashes at the start — 1 is decided? ✓

$C_{j+1} = (0,0,\ldots,1,1)$ — failure-free — 1 is decided

Rounds Lower Bound: Main Lemma

We consider only f-round executions such that:

- $f \le n-2$
- At most one process crashes in each round and at most f processes crash in each execution.
- In the round in which a process crashes, it sends messages to a prefix of processes, ordered by id's

Lemma: For any f-round execution α, α ≈ γ, where γ is the same as α during the first r rounds but has no crashes after round r, $0 \le r \le f$.

Rounds Lower Bound: Proof of Main Lemma (Base)

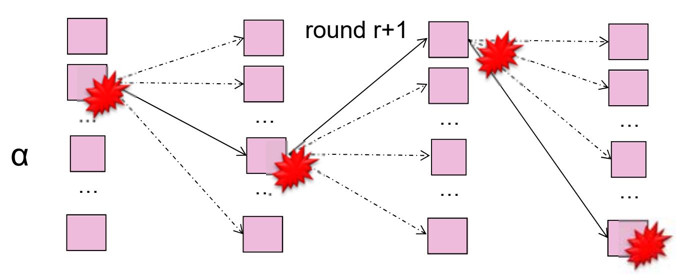
By backward induction on r

The base case, $r = f$, $\alpha = \gamma$
and the lemma is obvious

Lemma: For any f-round execution α, $\alpha \approx \gamma$, where γ is the same as α during the first r rounds but has no crashes after round r, $0 \le r \le f$.

Rounds Lower Bound: Proof of Main Lemma (Inductive Step)

Assume $r < f$ and that the lemma holds for r+1.

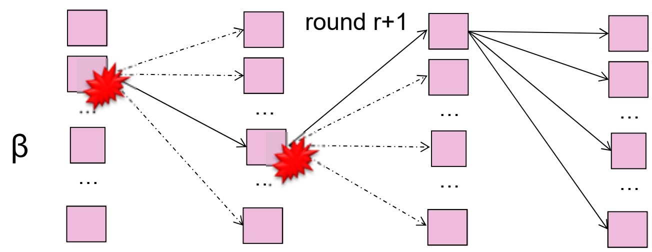


Rounds Lower Bound: Proof of Main Lemma (Inductive Step)

Assume r < f and that the lemma holds for r+1.

Let β be the same as α during its first r+1 rounds and has no crashes after round r+1

By induction, α ≈ β; we need to show β ≈ γ

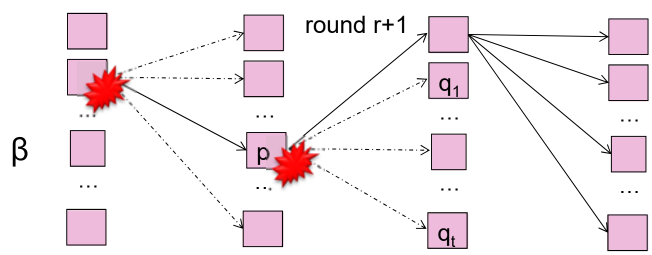


Rounds Lower Bound: Proof of Main Lemma (Inductive Step)

What happens in β?

p is the single process that crashes in round r+1 of β (if none fails then we are done)

$q_1, \ldots, q_t$ are the **correct** processes to which p does **not** send a message in round r+1 (in order of id's)

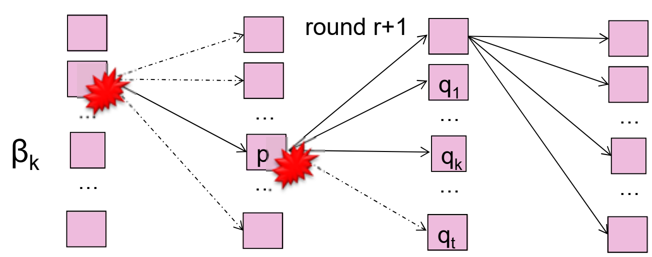


Rounds Lower Bound: Chain of Executions

β_k is the same as β in the first r+1 rounds, except that p sends messages to $q_1,\ldots,q_k$ in round r+1

$\beta_0 = \beta$

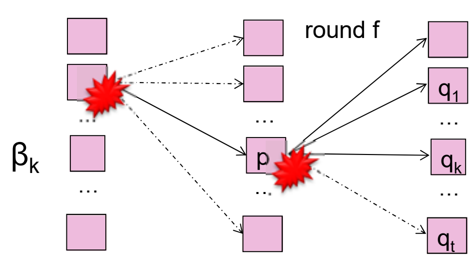
A correct process does not distinguish β_t from γ



Rounds Lower Bound: r = f-1

Some correct process ≠ q_k does not distinguish between β_k and β_{k-1} (there is one since f < n-2)

⇨ $\beta \approx \beta_t \approx \gamma$

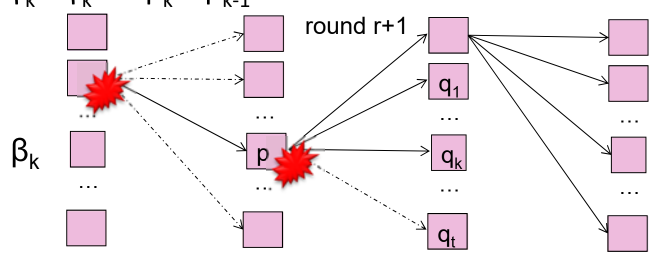


Rounds Lower Bound: $\beta_k \approx \beta_{k-1}$ for r < f-1

γ_k is the same as β_k for the first r+1 rounds, but q_k crashes in the beginning of round r+2 (cleanly) and there are no crashes after round r+2. By induction, $\beta_k \approx \gamma_k$

γ_k' is the same as β_{k-1} for the first r+1 rounds, but q_k crashes in the beginning of round r+2 (cleanly) and there are no crashes after round r+2. By induction, $\beta_{k-1} \approx \gamma_k'$

$\gamma_k \approx \gamma_k' \Rightarrow \beta_k \approx \beta_{k-1}$



Rounds Lower Bound: Completing the Proof

Theorem: Any consensus algorithm for n ≥ f+2 processes that tolerates f crashes requires ≥ f+1 rounds

Otherwise, apply initial configuration lemma

There is an initial configuration from which there are two executions α and γ that decide different values

In α and γ no processes crashes, except for one process that crashes before the start of γ

By previous lemma, α ≈γ

⇨ Same value is decided in both

Byzantine Failures

How Many Processes can Solve Consensus with One Byzantine Failure?

Validity: If all nonfaulty processes have input v, decide v

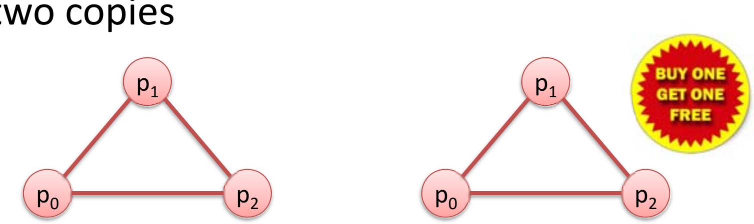
- Two processes?
  If p_0 has input 0 and p_1 has 1, someone has to change, but not both
  What if one processor is faulty?
  How can the other one know?
- Three processes?
  If p_0 has input 0, p_1 has input 1, and p_2 is faulty, then a tie-breaker is needed, but p_2 can act maliciously

Processes Lower Bound for f = 1

Theorem: Any consensus algorithm for one Byzantine failure must have at least four processes

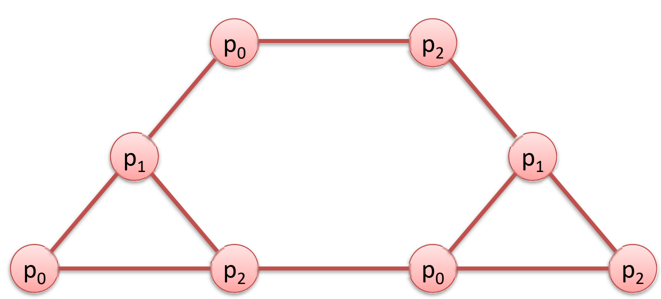
Suppose in contradiction there is a consensus algorithm for 3 processes and 1 Byzantine failure

Get two copies



Processes Lower Bound for f = 1

Rewire the copies

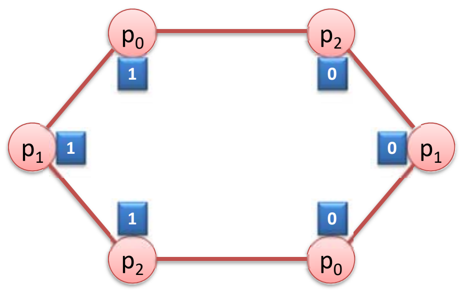


Processes Lower Bound for f = 1

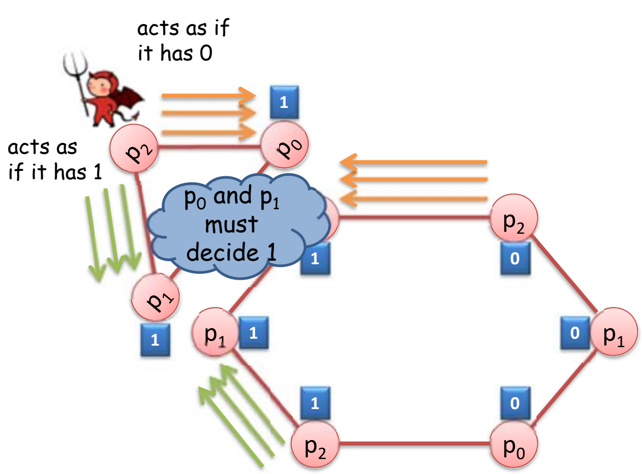
Rewire the copies and assign inputs

This execution does not have to solve consensus

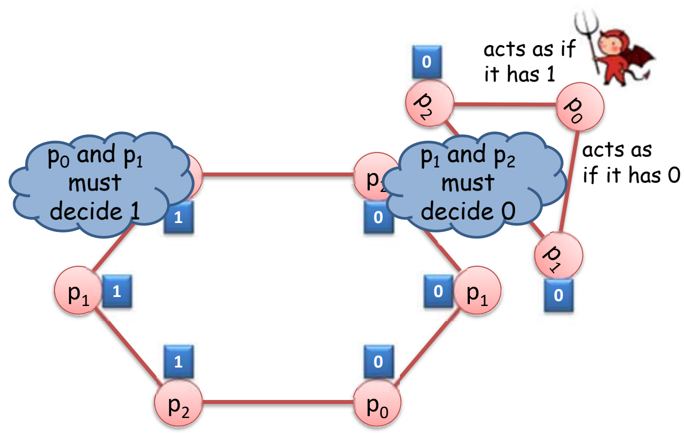
But it can **specify the behavior of faulty processes**



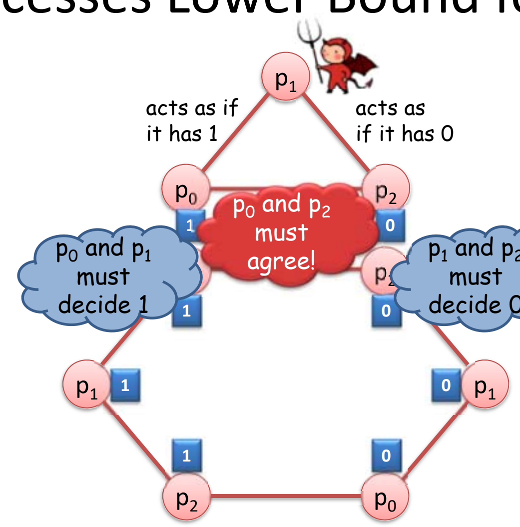
Processes Lower Bound for f = 1



Processes Lower Bound for f = 1



Processes Lower Bound for f = 1



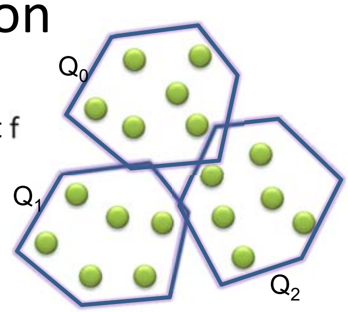
n > 3f for arbitrary f

Theorem: Any consensus algorithm for f Byzantine failures must have at least $3f+1$ processes

Proof by reduction to the 3:1 case

- Suppose in contradiction there is an algorithm $\mathcal{A}$ for $f > 1$ failures and $n = 3f$ total processes
- Use $\mathcal{A}$ to construct an algorithm for 1 failure and 3 processors, a contradiction

The Reduction



Partition the $n \leq 3f$ processes into three sets, Q_0, Q_1, and Q_2, each of size at most f

- p_0 simulates Q_0
- p_1 simulates Q_1
- p_2 simulates Q_2

If one process is faulty in the $n = 3$ system, then at most f processes are faulty in the simulated system

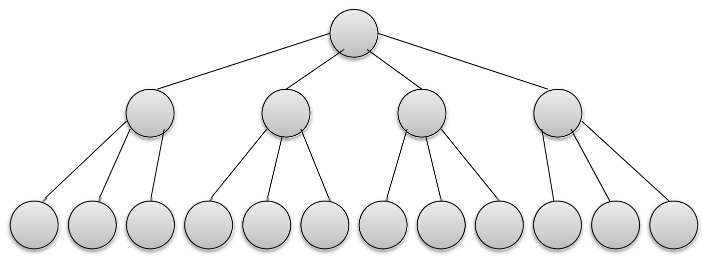
⇨ The simulated system is correct

Processes in the $n = 3$ system decide as the simulated processes

⇨ Their decisions are correct

Tree Algorithm

- This algorithm uses
  - *f + 1* rounds (optimal)
  - *n = 3f + 1* processors (optimal)
  - exponential size messages (very bad)
- Each process keeps a local tree data structure
- Values are filled in the tree during the *f + 1* rounds
- Then, the decision is calculated from the tree values



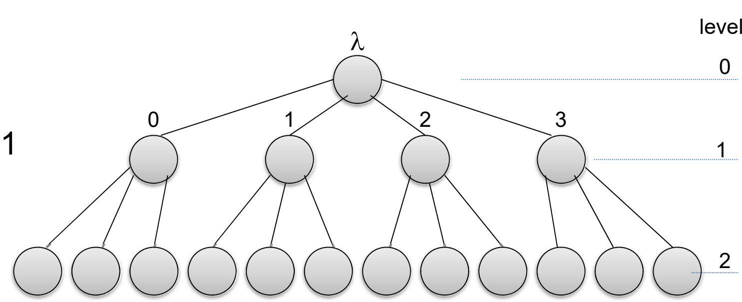
Local Tree Data Structure

Each node is labeled with a sequence of unique process identifiers

Root's label is the empty sequence λ; its level is 0

Root has n children, labeled 0 .. n - 1

n = 4, f = 1



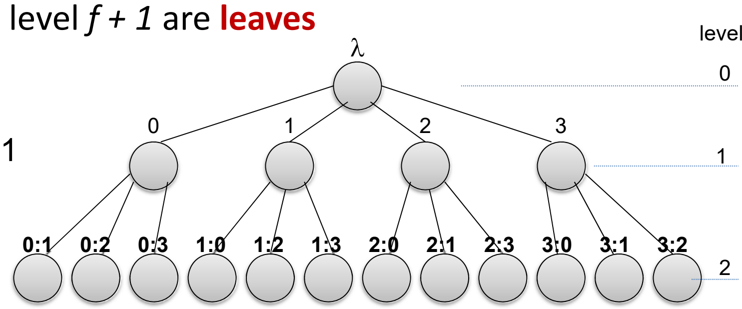
Local Tree Data Structure

Child node labeled *i* has *n - 1* children, labeled *i : 0* .. *i : n-1* (skipping *i : i)*

Node at level *d* with label *v* has *n - d* children, labeled *v : 0* .. *v : n-1* (skipping any index in *v*)

Nodes at level *f + 1* are **leaves**

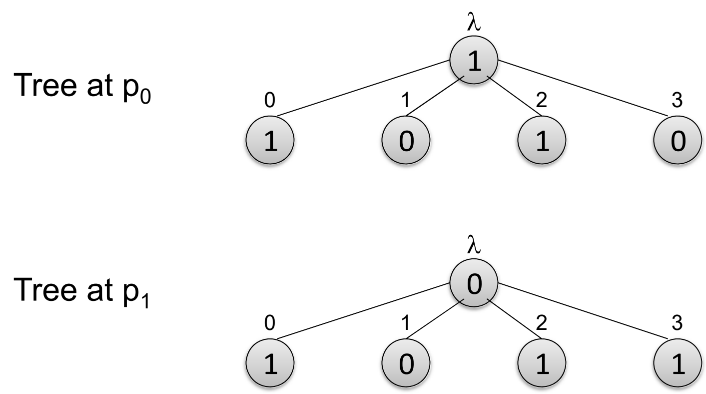
n = 4, f = 1



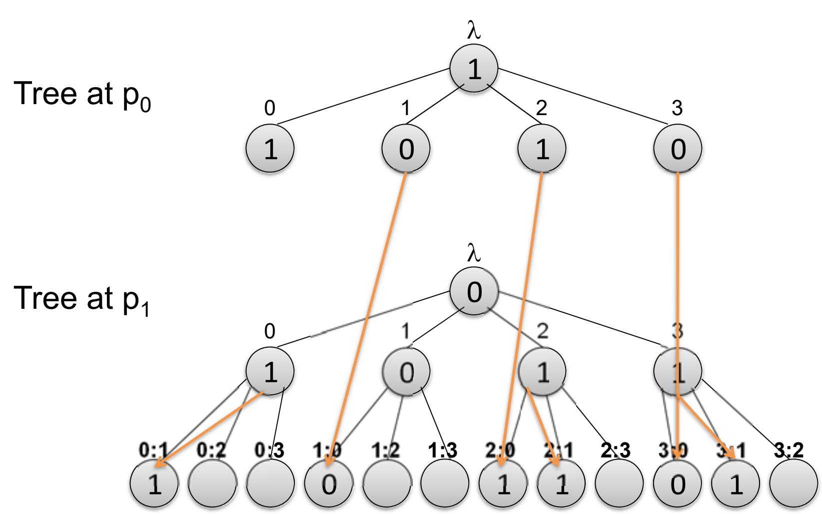
Filling in the Tree Nodes

- Initially store your input in the root (level 0)
- Round 1:
  - send level 0 of your tree to all
  - store value x received from each p_j in tree node labeled j (level 1); use a default if necessary
  - "p_j told me that p_j's input is x"
- Round 2:
  - send level 1 of your tree to all
  - store value x received from each p_j for each tree node k in tree node labeled k : j (level 2); use a default if necessary
  - "p_j told me that p_k told p_j that p_k's input is x"
- Continue for f + 1 rounds

Example Execution: Round 1

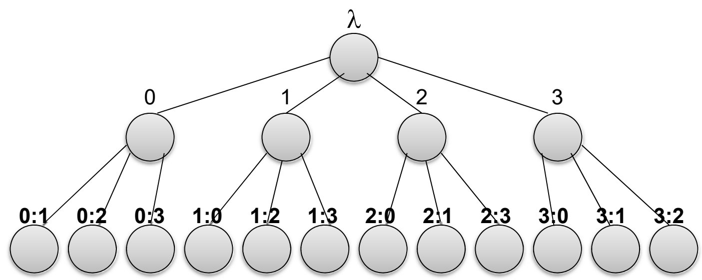


Example Execution: Round 2



Calculating the Decision

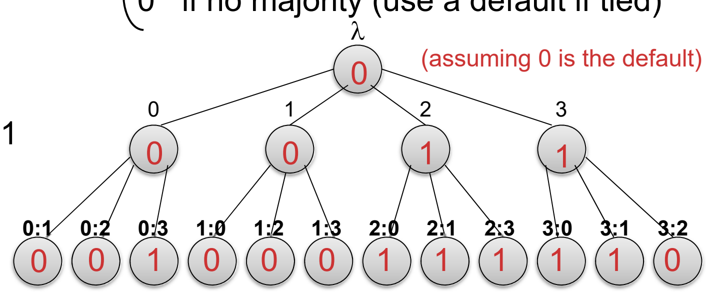
- In round $f + 1$, each process uses the values in its tree to compute its decision
- Recursively compute `resolve`(λ) for the root, based on the "resolved" values for its children



Calculating the Decision

`resolve`(π) =
- value in tree node labeled π if it is a leaf
- majority{`resolve`(π') : π' is a child of π}
  0 if no majority (use a default if tied)

n = 4, f = 1



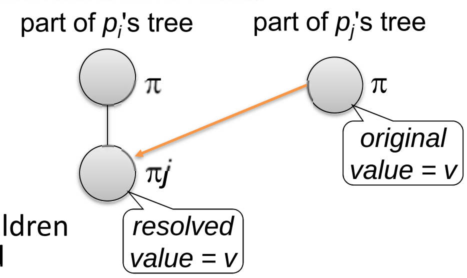
Resolved Values are Consistent

Lemma: If p_i and p_j are nonfaulty, then p_i's resolved value for tree node labeled πj (what p_j tells p_i for node π) equals what p_j stores in its node π

Proof by induction π's height (starting at the leaves)

By inductive hypothesis, resolved values for π children corresponding to nonfaulty processes are consistent

Since $n > 3f$ and π has $\geq n - f$ children majority of children correspond to nonfaulty processes

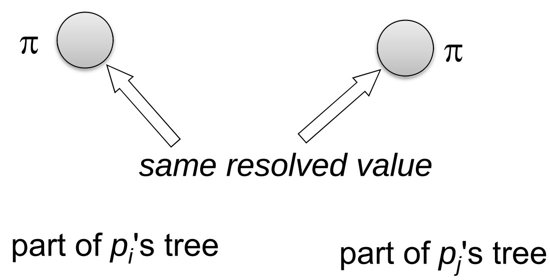


Resolved Values are Valid

- Suppose all nonfaulty processes have input v
- Nonfaulty process p_i decides `resolve`(λ), which is the majority among `resolve`(j), $0 \leq j \leq n\text{-}1$, based on p_i 's tree
- Since resolved values are consistent, resolve(j) (at p_i) is the value stored at the root of p_j's tree, which is p_j's input value if p_j is nonfaulty
- Since there is a majority of nonfaulty processes, p_i decides v

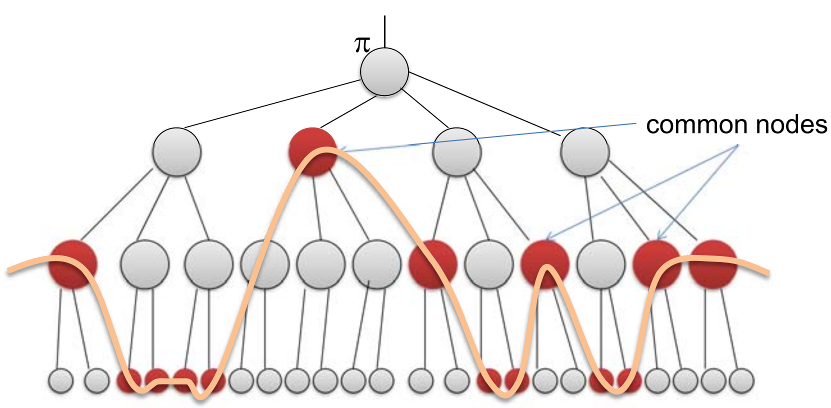
Common Nodes

A tree node π is **common** if all nonfaulty processes compute the same value of resolve(π)



Common Frontiers

A tree node π has a **common frontier** if there is a common node on every path from π to a leaf



Common Nodes and Frontiers

Lemma: If π has a common frontier, then π is common

Proof by induction on height of π, since resolve uses majority

Implies **agreement**:

- On each root-leaf path there is at least one node corresponding to a nonfaulty process
  - The nodes on the path correspond to f + 1 different processes
  - There are at most f faulty processes

⇨ This node is common (by consistency of resolved values)

⇨ The root has a common frontier

⇨ The root is common

Complexities of the Tree Algorithm

- $n > 3f$ processors
- $f + 1$ rounds
- exponential size messages:
  - each message in round r contains $n(n\text{-}1)(n\text{-}2)\dots(n\text{-}(r\text{-}2))$ values
  - When $r = f + 1$, this is exponential if f is more than a constant relative to n

A More Efficient Algorithm?

Better message complexity by increasing the number of rounds and ratio of nonfaulty processes

- n > 4t, 2(f + 1) rounds

t = max # of failures
f = actual # of failures

Aside: there are algorithms with

- Polynomial number of message bits
- f+1 rounds
- n > 3t

Yoram moses

Phase King Algorithm (n > 4t, 2(f+1) rounds)

Code for process p_i

```
pref = my input
first round of phase k, 1 ≤ k ≤ f+1:
  send pref to all
  receive prefs of others
  let maj be value that occurs > n/2 times // default 0
  let mult be number of times maj occurs

second round of phase k:
  if i = k then send maj to all  // I am the phase king
  receive tie-breaker from p_k              // default 0
  if mult > n/2 + f then
      pref := maj
  else
      pref := tie-breaker
  if k = f + 1 then decide pref
```

Unanimous Phase Lemma

Lemma: If all nonfaulty processes prefer v at start of phase k, then all prefer v at end of phase k

Since $n > 4f$, it follows that $n - f > n/2 + f$

Therefore, if all nonfaulty processes have input v

⇨ At start of phase 1, all nonfaulty processes prefer v

⇨ At end of phase 1, all nonfaulty processes prefer v

⇨ At start of phase 2, all nonfaulty processes prefer v

⇨ At end of phase 2, all nonfaulty processes prefer v

⇨ …

⇨ At end of phase $f + 1$, all nonfaulty processes prefer v and decide v

Nonfaulty King Lemma

Lemma: If p_k is nonfaulty, then all nonfaulty processes have same preference at end of phase k

Proof: If two nonfaulty processes p_i and p_j use p_k's tie-breaker, they have same preference

If p_i uses a majority value v and p_j uses p_k's tie-breaker then p_k majority value is also v

If both p_i and p_j use their majority value, then it must be the same value

Agreement in Phase King Algorithm

f + 1 iterations ⇨ at least one with a nonfaulty king

Nonfaulty King Lemma ⇨ at the end of that phase, all nonfaulty processes have same preference

Unanimous Phase Lemma ⇨ from that phase on, all nonfaulty processes have same preference

⇨ All nonfaulty processes decide on the same value

Phase Queen Algorithm (n > 3t, 3(f+1) rounds)

Code for process p_i

```
pref = my input
first round of phase k, 1 ≤ k ≤ f+1:
  send pref to all
  receive pref's of other processes
  pref = abort
  if some value v appears ≥ n-f times then pref = v
second round of phase k, 1 ≤ k ≤ f+1:
  send pref to all
  receive pref's of others
  if some value v appears mult > f times then
      pref = smallest such v        // abort is largest
third round of phase k, 1 ≤ k ≤ f+1:
  if i = k then send pref           // I am phase queen
  receive tie-breaker from k
  if ( pref = abort or mult < n-f )
            and (tie-breaker ≠ abort)
      then pref = min(1,tie-breaker)
```

Unanimous Phase Lemma

Lemma: If all nonfaulty processes prefer *v* at start of phase *k*, then all prefer *v* at end of phase *k*

For each phase *k*:

- At the end of the first round, the value of *pref* for all nonfaulty processes, is v or abort, for some v $\in$ {0,1}
- At the end of the second round, the value of *pref* for all nonfaulty processes, is v or abort, for the same v

Nonfaulty Queen Lemma

Lemma: If p_k is nonfaulty, then all nonfaulty processes have same preference at end of phase *k*

- All nonfaulty processes accept the phase king's message
- Some nonfaulty process ignores the king since mult ≥ *n-t.* Then mult > ***f*** for every nonfaulty process , and its pref is the same.

After this phase, Unanimous Phase Lemma ensures agreement is maintained until the algorithm terminates

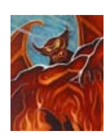
Randomized Consensus

- Weaken the termination condition and measure the expected time to termination
- Agreement and validity remain the same
- Allow to overcome the asynchronous impossibility and the synchronous lower bound (we'll see only the first)

Two Sources of Nondeterminism

- In a **randomized algorithm**, processes flip coins to determine their next steps 
  - Several possible executions
- But **even a deterministic algorithm** has several possible executions (from a fixed input)
  - Due to asynchrony and/or failures

- Separate the latter under the control of an **adversary**
  - Determines the next event to occur after an execution prefix
  - Must obey admissibility conditions according to model
  - May have other limitations (what information it can observe, how much computational power it has)



Evaluating a Distributed Randomized Algorithm

- An execution of a specific algorithm, exec(C_0,R,$\mathcal{A}$), is uniquely determined by
  - an initial configuration C_0
  - a sequence of random numbers R
  - an adversary $\mathcal{A}$
- Given a predicate *Pred* on executions, a fixed adversary $\mathcal{A}$ and an initial configuration C_0

  Pr[*Pred*] = Prob {R : exec(C_0,R,$\mathcal{A}$) satisfies *Pred*}
- Let T be a random variable (time)

  $$\exp(T, \mathcal{A}, C_0) = \sum_t t \Pr[T = t]$$

Expected Time Complexity of a Randomized Distributed Algorithm

The **expected time complexity** is the max over all admissible adversaries $\mathcal{A}$ and initial configurations C_0, of the expected time for that particular $\mathcal{A}$ and C_0

$$\max_{\text{adversary } \mathcal{A},\ \text{initial configuration } C_0} \exp(T(\text{Alg}, , \mathcal{A}, C_0))$$

Worst-case average: for the worst adversary (asynchrony and failures) and initial configuration, average over the random choices of the algorithm

Extend naturally to other measures (like RMRs)

Structure of Consensus Algorithm

The algorithm has two components

- **Phase-based voting scheme** using individual processors' preferences to reach agreement (when possible)
  We use **extended adopt-commit**
- A **shared coin procedure** used to break ties among these preferences

Shared Coin

A **shared coin** with **agreement probability** ρ (with no input) returns a binary output, s.t.

- For every $v \in \{0,1\}$, all nonfaulty processors executing the procedure **output v with probability at least ρ**

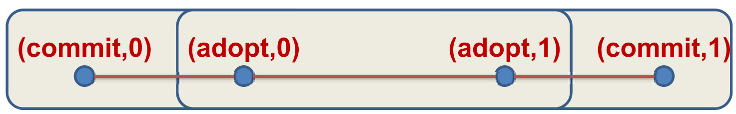
A simple and very resilient shared coin with $\rho=1/2^n$ bias is when each process outputs a (uniform) random bit

There are more sophisticated constructions

Recall: Adopt-Commit

The decision is **(grade, y_i)**,
where grade is either **adopt** or **commit**, such that

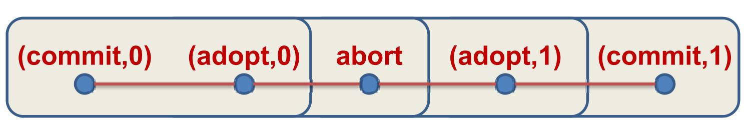
- **Graded agreement**: if a process decides **(commit,y_i)** then all processes decide **(adopt,y_i)** or **(commit,y_i)**
- **Validity: y_i** was proposed by some process
- **Convergence:** If only **y_i** is proposed before p outputs **(grade,y_i)** then **grade = commit**
- **Termination:** A process returns within a finite number of steps



Extended Adopt-Commit

The decision is either $\perp$ (**abort**) or **(grade, y_i)**,
where grade is either **adopt** or **commit**, such that

- **Graded agreement**: if a process decides **(commit,v)** then all processes decide **(adopt,v)** or **(commit,v)** and if a process decides **(adopt,v)** then no process adopts a different value
- **Validity**: if all nonfaulty processes propose **v** then all nonfaulty processes return **(commit,v)**
- **Termination:** A process returns within a finite number of steps



Implementing Extended Adopt-Commit w/ Byzantine Failures

- Assumes $n > 3f$
- *2* (asynchronous) rounds

```
send v to all
receive values from others
let maj be value that occurs > n/2 times (0 if none)
let mult be number of times maj occurs
if mult ≥ n-f then send maj to all

receive values from others
let maj' be value that occurs most times
let mult' be number of times maj' occurs
if mult' ≥ n-f return (commit,maj')
else if mult' ≥ f+1 return (adopt,maj')
else return abort
```


Randomized Consensus w/ Extended Adopt-Commit

Assume we have a shared coin algorithm with agreement probability ρ and time complexity T_{coin}

```
pref = my input
Phase k
(grade,v) = Extended-adopt-commit(pref) ] T_EAC
flip = Shared-coin() ] T_coin
if grade == abort
      pref = flip
if grade == adopt
      pref = v
else                                // grade == commit
      decide v                      // but continue to echo
```

Time complexity of a phase is (T_{EAC}+T_{coin})

Validity

Unanimous Phase Lemma: If all nonfaulty processes prefer *v* at start of phase *k*, then all do at end of phase *k*

If all processes have input *v* ⇨ all prefer *v* in phase 1

By the lemma (and graded agreement),
all nonfaulty processes decide *v* in phase 1

Agreement

Lemma: If p_i decides *v* in phase *r*, then all nonfaulty processes decide *v* by phase *r + 1*

Proof: Let r be the earliest phase in which a process (say, p_i) decides (say, on v)

p_i got (commit,v) in phase r

All other processes got (adopt,v) in phase r, so they prefer v in phase r+1 and by previous lemma, decide v

Termination

Lemma: The probability that all nonfaulty processes decide in a phase is at least ρ

Proof: If all nonfaulty processes set their preference in phase r using Shared-coin

- With probability 2ρ, they all get the same value (ρ for 0 and ρ for 1); lemma follows from unanimous phase lemma

If some processes **do not** set their preference using Shared-coin

- All of them have the save value v as phase r preference
- With probability $\geq \rho$, all processes get v from Shared-coin

Expected Number of Phases

Probability of all deciding in any given phase $\geq \rho$

⇨ Probability of terminating after *i* phases is $(1-\rho)^{i-1}\rho$

⇨ Number of phases until termination is a geometric random variable whose expected value is $1/\rho$

The time complexity of the algorithm is $\rho^{-1}(T_{EAC}+T_{coin})$,
T_{EAC} is the time complexity of Extended Adopt-Commit
T_{coin} is the time complexity of Shared-coin

Better Shared Coin

Back to shared memory and crash failures…

- constant agreement probability ρ
- polynomial total number of steps T_{coin}

```
Shared SumCoins[i], NumFlips[i], initially 0

while ()
      c = random(-1,+1)
      SumCoins[i] += c  // written only by i, atomic
      NumFlips[i]++     // written only by i, atomic
      read NumFlips[0,..,n-1]
      if Σ NumFlips[0,..,n-1] > n²
            read SumCoins[0,..,n-1]
            return( sign( Σ SumCoins[0,..,n-1] ))
```

simpler than (0,1)

Step Complexity

- Number of coins flipped (= iterations of the while loop) $< n^2+n$
- $O(n)$ steps per iteration ⇨ $O(n^3)$ total work

```
Shared SumCoins[i], NumFlips[i], initially 0

while ()
      c = random(-1,+1)
      SumCoins[i] += c  // written only by i, atomic
      NumFlips[i]++     // written only by i, atomic
      read NumFlips[0,..,n-1]
      if Σ NumFlips[0,..,n-1] > n²
            read SumCoins[0,..,n-1]
            return( sign( Σ SumCoins[0,..,n-1] ))
```

Agreement Parameter

- Among t^2+t independent unbiased coins, the minority is less than $t^2/2$ with probability > ½
- Probability all processes get same value > ¼

```
Shared SumCoins[i], NumFlips[i], initially 0

while ()
      c = random(-1,+1)
      SumCoins[i] += c   // written only by i, atomic
      NumFlips[i]++      // written only by i, atomic
      read NumFlips[0,..,n-1]
      if Σ NumFlips[0,..,n-1] > n²
            read SumCoins[0,..,n-1]
            return( sign( Σ SumCoins[0,..,n-1] ))
```


Space Lower Bound

$\Omega(n)$ registers are necessary for nondeterministic solo-terminating consensus using reads and writes

[Leqi Zhu, 2016]

Use nondeterminism to capture randomization

☞ Multiple solo executions (of the same process) from a specific configuration

Proof very similar to Ω(n) mutex space l.b.

Recall Valence w/ Small Twist

For a configuration C, and processes p_i and p_j

- p_i is ***v-solo-potent*** in C if
  p_i can decide v in some solo execution from *C*
- p_i is ***v-solo-univalent*** in C if p_i is v-solo-potent but not $\bar{v}$-solo-potent in C (**solo-univalent** in general)
- p_i and p_j are **solo-bivalent** in C if
  p_i is v-solo-potent in C and p_j is $\bar{v}$-solo-potent in C

p_0 and p_1 are solo-bivalent in some initial configuration

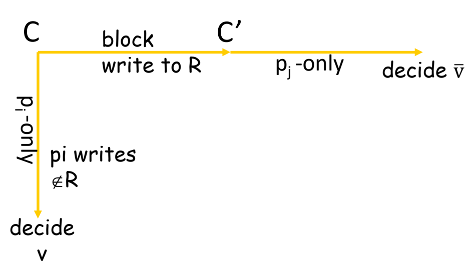
Follows by a standard proof

Also, Recall Covering

A **process covers a register** R in a configuration C if it is about to write to R in C

Extend to **a set of k processes covering k registers**

Block write: all processes write



Assume processes P cover registers R in a configuration C.
Let C' be the configuration after their block write from C,
and assume process $p_j \neq p_i$ is $\bar{v}$-potent in C'.
If a process $p_i \notin P$, $p_j \neq p_i$, is *v*-solo-potent in C,
then **p_i writes to a register $\notin$R in its solo execution**

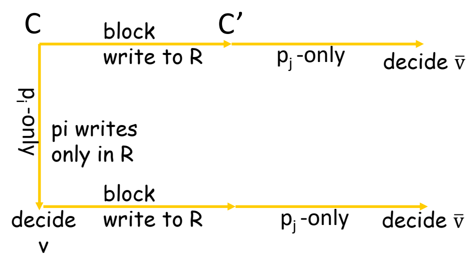
Proof by Contradiction

If p_i writes only to R

Apply the solo execution of p_j after the block write

p_i still decides $\bar{v}$

⇨ Contradiction to agreement



Assume processes P cover registers R in a configuration C.
Let C' be the configuration after their block write from C,
and assume process $p_j \neq p_i$ is $\bar{v}$-potent in C'.
If a process $p_i \notin P$, $p_j \neq p_i$, is v-solo-potent in C,
then **p_i writes to a register $\notin$R in its solo execution**

Key Lemma

If p_0 and p_1 are solo-bivalent in a configuration C then there is a $\{p_0, \ldots, p_k\}$-only execution from C ending in C' s.t. **p_0 and p_1 are solo-bivalent in C'** and **$p_2, \ldots, p_k$ cover k-1 different registers in C'**

Apply the lemma with $k = n - 1$, starting from an initial configuration in which p_0 and p_1 are solo-bivalent

⇨ $n - 2$ space lower bound

Can be improved to $n - 1$

Help Lemma

If p_0 and p_1 are solo-bivalent in C and processes P (other than p_0 and p_1) cover a set of registers R, then **p_0 and p_1 are solo-bivalent after a solo execution by either p_0 or p_1, followed by a block write to R**

p_0 and p_1 are solo-bivalent $C \xrightarrow{p_0\text{-only or } p_1\text{-only}} \xrightarrow{\text{block write to R}} C'$ p_0 and p_1 are solo-bivalent

Proof of Help Lemma

If p_0 and p_1 are solo-bivalent in C and processes P (other than p_0 and p_1) cover a set of registers R, then **p_0 and p_1 are solo-bivalent after a solo execution by either p_0 or p_1, followed by a block write to R**

p_0 and p_1 are solo-bivalent $C \xrightarrow{\text{block write to R}} C'$ p_0 and p_1 are solo-bivalent ✓

Proof of Help Lemma

p_0 and p_1 are solo-bivalent C block write to R C' p_0 and p_1 both v-solo-univalent

p_0 and p_1 are solo-bivalent C block write to R C' p_0 and p_1 are solo-bivalent ✓

Proof of Help Lemma

p_0 and p_1 are solo-bivalent C block write to R C' p_0 and p_1 both v-solo-univalent

maximal

block write to R D p_0 and p_1 both v-solo-univalent

block write to R D' p_1 is v-solo-univalent

p_0-only

decide $\bar{v}$

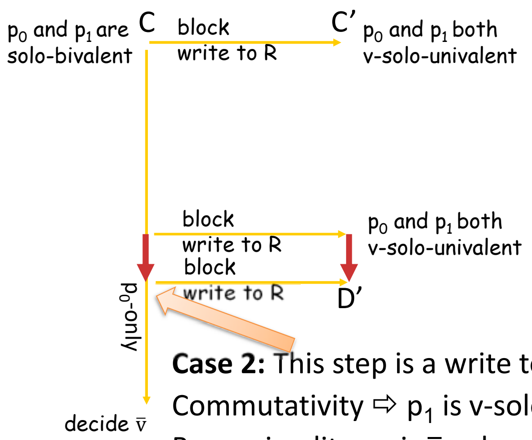
Case 1: This step is a read or a write to a register $\in$ R

p_1 does not distinguish D and D' ⇨ p_1 is v-solo-univalent in D'

By maximality, p_0 is $\bar{v}$-solo-potent in D'

⇨ p_0 and p_1 are solo-bivalent in D' ✓

Proof of Help Lemma



Case 2: This step is a write to a register $\notin$ R

Commutativity ⇨ p_1 is v-solo-univalent in D'

By maximality, p_0 is $\bar{v}$-solo-potent in D'

⇨ p_0 and p_1 are solo-bivalent in D'

Back to Proving the Key Lemma

If p_0 and p_1 are solo-bivalent in a configuration C then there is a $\{p_0, \ldots, p_k\}$-only execution from C ending in C' s.t. **p_0 and p_1 are solo-bivalent in C'** and **$p_2, \ldots, p_k$ cover k-1 different registers in C'**

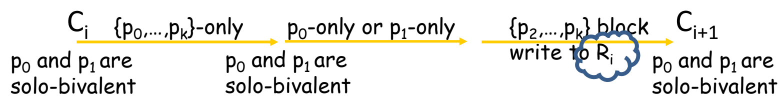
By induction on k, with a trivial base case k=1

For the induction step (k+1)

Repeated apply induction hypothesis & help lemma

Back to Proving the Key Lemma

If p_0 and p_1 are solo-bivalent in a configuration C then there is a $\{p_0, \ldots, p_k\}$-only execution from C ending in C' s.t. p_0 and p_1 are solo-bivalent in C' and $p_2, \ldots, p_k$ cover k-1 different registers in C'

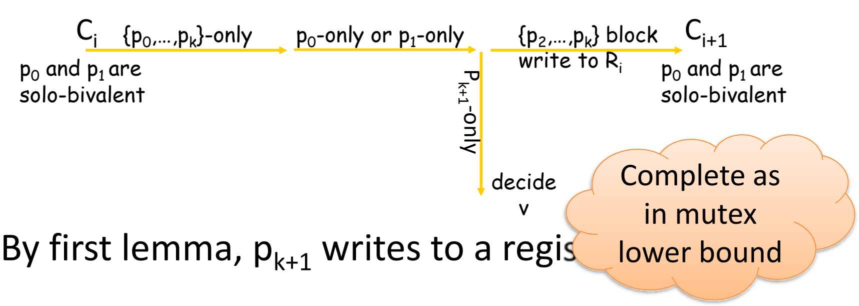


Repeated apply induction hypothesis & help lemma

For some i < j, $R_i = R_j$

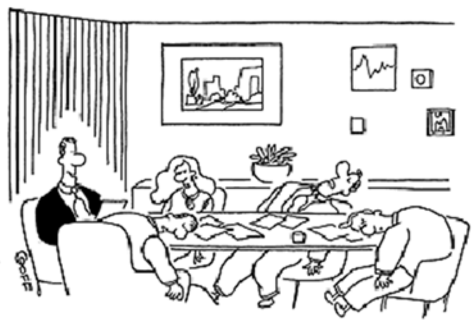
Back to Proving the Key Lemma

If p_0 and p_1 are solo-bivalent in a configuration C then there is a $\{p_0, \ldots, p_k\}$-only execution from C ending in C' s.t. p_0 and p_1 are solo-bivalent in C' and $p_2, \ldots, p_k$ cover k-1 different registers in C'



By first lemma, p_{k+1} writes to a regis

Finale



"At last we've reached a consensus!
This meeting is boring!"